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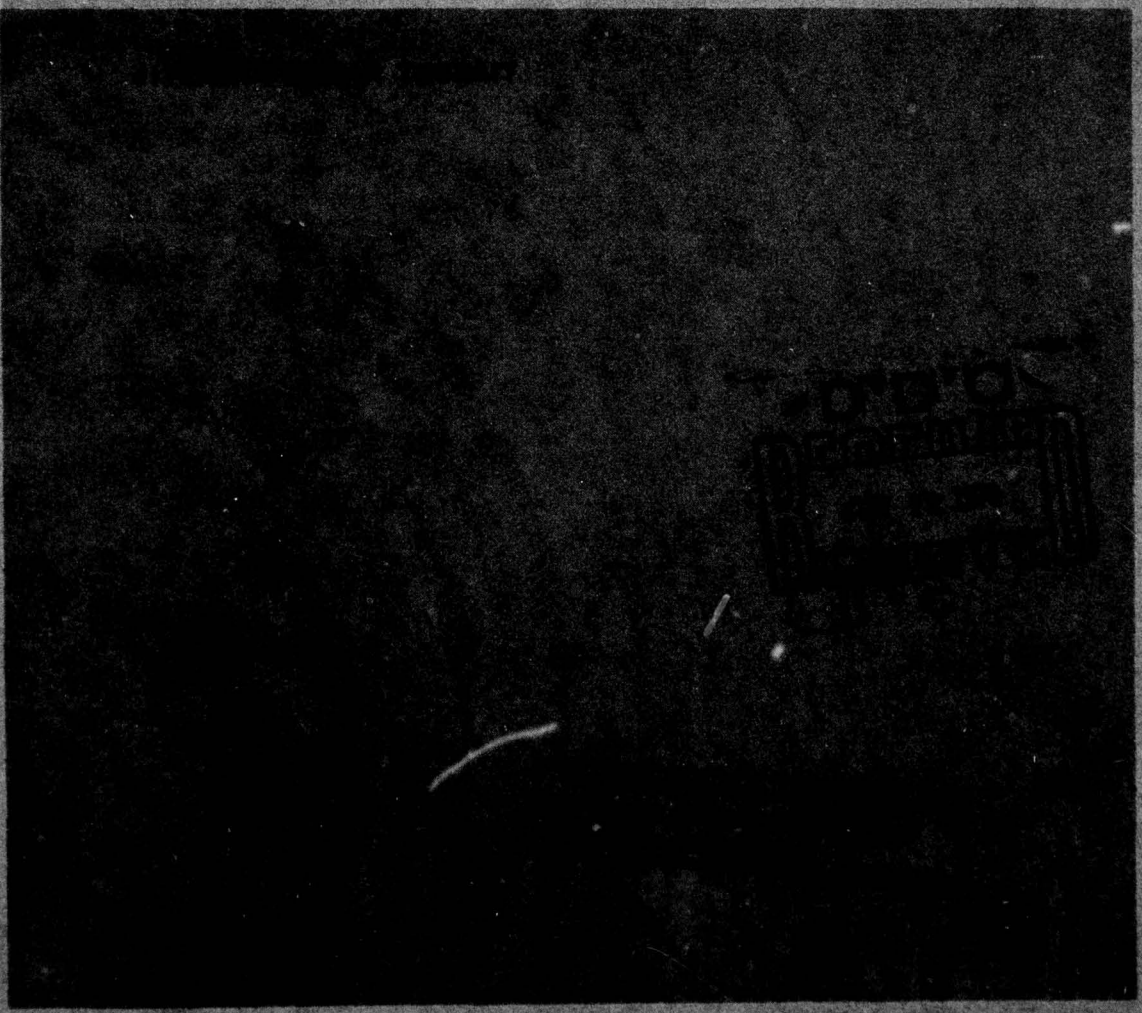
ESTIMATION FOR UNCERTAIN LINEAR SYSTEMS WITH JUMP PARAMETERS



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The problem of state estimation for a class of linear discrete-time dynamical systems with unknown time-varying parameters is investigated. Attention is focused mainly on systems with unknown noise statistics. Two different approaches to modelling and estimation under time-varying uncertainties are investigated. In one of the approaches, a finite state Markov chain model is used for the jump parameters which can take values only from a finite set with transitions from one value to another determined by a Markov transition probability matrix. The transition matrix may or may not be known; if unknown, → next page

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20. ABSTRACT (continued)

it is assumed to belong to a finite set. A Bayes optimal solution is obtained in a recursive form and several suboptimal algorithms are discussed to alleviate the large storage and computation requirements of the optimal estimator. The asymptotic behavior of the optimal solution for the case of unknown transition probabilities is analyzed. Numerical examples are presented in support of the analyses. In the second approach, multiple bounds on the unknown parameter values and its time derivatives are assumed to be available. A detection-estimation approach is proposed for state estimation and its asymptotic behavior is analyzed. The main objective of the proposed approach is to reduce the pessimism of the standard minimax estimator for large observation records and large uncertainties, while retaining its desirable small-sample properties. The performance of the scheme is illustrated by means of simulation examples.

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by

Jitendra Kumar Tugnait

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ESTIMATION FOR UNCERTAIN LINEAR SYSTEMS WITH JUMP PARAMETERS

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THESIS

Submitted in partial fulfillment of the requirements
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in the Graduate College of the
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University of Illinois at Urbana-Champaign, 1978

ABSTRACT

The problem of state estimation for a class of linear discrete-time dynamical systems with unknown time-varying parameters is investigated. Attention is focused mainly on systems with unknown noise statistics. Two different approaches to modelling and estimation under time-varying uncertainties are investigated. In one of the approaches, a finite state Markov chain model is used for the jump parameters which can take values only from a finite set with transitions from one value to another determined by a Markov transition probability matrix. The transition matrix may or may not be known; if unknown, it is assumed to belong to a finite set. A Bayes optimal solution is obtained in a recursive form and several suboptimal algorithms are discussed to alleviate the large storage and computation requirements of the optimal estimator. The asymptotic behavior of the optimal solution for the case of unknown transition probabilities is analyzed. Numerical examples are presented in support of the analyses. In the second approach, multiple bounds on the unknown parameter values and its time derivatives are assumed to be available. A detection-estimation approach is proposed for state estimation and its asymptotic behavior is

analyzed. The main objective of the proposed approach is to reduce the pessimism of the standard minimax estimator for large observation records and large uncertainties, while retaining its desirable small-sample properties. The performance of the scheme is illustrated by means of simulation examples.

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CHAPTER 1

INTRODUCTION

1.1. General

This thesis is concerned with the estimation problem for a class of linear discrete-time dynamical signal models with unknown time-varying parameter values. The problem of estimating the state variables of a dynamical system, given noisy observations of the output variables, is of fundamental importance in control and communication theory. In control problems the final goal is often to design control strategies for a particular system, which requires the availability of the state of the controlled plant. In communication theory problems, the primary interest is estimation of the state of the system itself.

When the models for the signal and the noise are completely specified, it is possible, at least theoretically, to obtain optimal solutions to the problem of state estimation under various optimality criteria [65,28]. The problem is considerably more difficult when uncertainty exists regarding the system parameters, the system model, or the noise statistics; especially if the uncertain quantities are time-varying. An extensive amount of work has been done on the problem of state estimation and identification for stationary Gaussian signal models. A popular model is the state space representation where a linear time-invariant system is driven by and observed in white Gaussian noise. The main advantage of this model is that it is analytically tractable while also being not too far removed from "physical reality".

In this thesis we are concerned with a class of linear discrete-time dynamical stochastic systems with time-varying parameters. Attention is focused mainly on systems with unknown time-varying noise statistics. The motivations for such an assumption are as follows. First, there exists a large class of physical phenomena which can be modeled in terms of linear systems driven by (and/or observed in) noise with (unknown) time-varying statistics. Second, quite often we are interested in approximating a higher order system with a lower order model (due to, e.g., limitations on available computation and storage capabilities), or in approximating a nonlinear model with its linearized equivalent. The error in modeling can then be thought of as a noise process (possibly white and/or Gaussian) whose statistics might change with time depending upon the approximation of the chosen model to the original system. The objective then becomes how to choose the noise statistics to make the approximation best possible in some sense. We also address the problem of state estimation under uncertain observations. The unknown parameters will be assumed to switch among a set of finite number of values; hence the use of the terms jump or switching parameters.

Two different approaches to state estimation for systems with jump parameters are investigated. In one of the approaches, the jump parameters are assigned a probabilistic description. We assume that the parameters can take values only from a finite set, with transitions from one value to another determined by a Markov transition probability matrix. The set of values taken by the parameters is assumed to be known. However, the transition matrix may be unknown and is assumed to belong to a finite

set. A Bayesian approach for optimal state estimation may be appropriate here. The second approach is used only for the case of unknown time-varying noise statistics. It is assumed that a statistical description is not appropriate in this case; rather multiple bounds (disjoint or nested) are assumed to be available on the unknown parameter values and its time derivative(s). A combined detection-estimation approach is appropriate here, and is based, to an extent, on the approach given in [45,46]. This approach is essentially a heuristic extension of the standard minimax scheme to the case when multiple bounds on the unknown parameters are available.

The model of linear discrete-time dynamical system with jump parameters can be useful in many problems of interest. The Markov jump process can be used as a model for periodic but random step changes in system parameters (or noise statistics) where the measurement of such changes is not feasible, either for technical or economic reasons. A major advantage of this model is that it allows some nonstationary environments with random properties to be treated by techniques similar to those used for stationary Gaussian disturbances. To further motivate the choice of this switching model, consider the following example with switchings in the noise statistics. Let a hypothetical submarine be constrained to randomly maneuver only in depth. As a first approximation assume that the target (the submarine) maneuvers only at discrete times, known to the tracking vehicle. The submarine will have a continuous range in which to choose its next depth. In modelling the variations

in depth to which the submarine can descend, certain discrete depths (states) $d_1, d_2, \ldots, d_n$ are chosen. These n states act as mean values around which the depth of the target randomly varies. Statistically modelling the maneuvering target, we assume that the transitions among the n states can be described by a Markov chain. Both rapid and slow maneuvering of the target can be modelled well by suitably selecting the transition matrices. Such a model for maneuvering targets has been used in [62,70]. Moose [43] used a more general model of a semi-Markov process. In [12], switchings in the driving noise covariance are used to model variable maneuverability of an airborne target. The Markov jump process model may also be used in problems such as estimation for systems subject to possible component/subsystem failures [1,2,69], and certain control applications [1,48,59-61]. We also note that the detection-estimation approach may be used in the case of above-mentioned examples if a statistical description of the switching parameters is either inappropriate or unavailable.

Finally, the important case of systems with uncertain observations may be modelled by expressing the interrupted observation mechanism in terms of a two-state Markov chain [44,51-53]. Such a situation may be caused by an intermittent failure in the observation mechanism. It may also occur in a tracking situation when sensor returns may originate from something other than a single object being tracked, such as other objects being

tracked, new objects not yet being tracked, false alarms, clutter, and radio frequency interference [56]. Furthermore, there are situations where the data acquisition process is often interrupted physically or artificially; and a continuous supply of information about the state of the system is impossible. For example, there exists a class of control systems where the observations are not available at every instant due to either the physical impossibility or the cost in taking observations [4].

1.2. Historical Survey

As remarked earlier an extensive amount of work has been done on the problem of state estimation and identification for stationary Gaussian models. When the signal and the noise models are completely known, well-known solutions exist [28,65]. A tutorial introduction to the problem of identification for stationary Gaussian models is available in the recent text by Goodwin and Payne [22]. An extensive survey of the work in identification may be found in Astrom and Eykoff [6]; brief surveys are available in Mehra [38] and Pearson [47]. More recent results may be found in [40] and [1974 Dec., IEEE Transactions on Automatic Control - Special Issue].

Existing approaches to state estimation for stationary Gaussian models with unknown parameters can be roughly classified into two major categories: minimax estimation [17,47,55] and adaptive estimation [6,7,34,22,38,39]. A minimax solution to an estimation problem minimizes a given cost function for the worst case values of the unknown parameters which are assumed to belong to a compact set. The minimax schemes

yield estimates which are too conservative when large observation records are available but are robust for small samples. Also, if other than the worst case values of the parameter occur, the performance of the minimax estimator usually deteriorates. A number of approaches can be taken to adaptive estimation. Some of these are: Bayesian estimation, maximum likelihood (ML) estimation, correlation methods, covariance matching techniques, and prediction error methods. In the Bayesian estimation schemes [24,34,36], the unknown parameters are characterized as random variables with known probability distribution and then the Bayesian minimum mean-squared error (MMSE) state estimate is found. In the maximum likelihood estimation schemes, first the unknown parameters are estimated using the maximum likelihood techniques and then these estimates are used in the proper state estimator algorithm. Such schemes have good large sample properties in that the state estimate converges to the optimal estimate (corresponding to the completely known model case) as the length of the observation record tends to infinity. The Bayesian scheme also has this large sample property in addition to also being optimal at every time instant. For small samples, the maximum likelihood schemes usually yield state estimates having large error covariances. The other adaptive schemes mentioned before are quite similar to the maximum likelihood schemes; they differ from the ML techniques only in that different techniques are used to estimate the unknown parameter values.

Both the above-mentioned classes of approaches are applicable mainly when the uncertainties are time-invariant. To extend these techniques to the time-varying uncertainties case, one must provide some description of how the parameters vary with time. For example, in [11], it is assumed

that a state-space model, driven by white noise, of the nonstationary covariance parameters is available. Similar assumptions have been made in [9]. In [7], a fading memory filter is proposed to track the time variations in the unknown noise covariances. All these extensions are quite ad-hoc, and need on-line "tuning." Finally, a combined detection-estimation approach has also been used [33,41,21,46,55]. It is particularly suited to cases where information regarding the uncertainties is available in the form of bounds (multiple hypotheses). So far, this approach has been found to be intractable for the case of time-varying uncertainties.

One of the objectives of this thesis is to investigate alternative models for characterizing time-variations in the system parameters. The objective is to choose a model which is general enough to describe a large class of time-varying characteristics of the system parameters, while, at the same time, is also analytically and computationally tractable. The Markov jump process is one such model for time-varying parameters, and is a statistical description of the time-variation. The bounding sets on the values of unknown (time-varying) parameters and their derivative(s) represent a deterministic model.

The problem of state estimation in switching environments (i.e., Markov dependent switchings in noise statistics) when the transition probability matrix is known, has been treated in [1,2,67] and several other papers. They differ primarily in the types of approximation to the optimal solution, which is difficult to compute because of exponentially increasing computation and storage requirements with time. The problem

of state estimation for systems with interrupted observations has been treated in [26,27,44,51-53]. It is similar to state estimation in switching environments. Previous work on state estimation in switching environments has not investigated the problem of unknown transition matrices. A much less general version of the problem was considered by Kashyap [31]. He considered identification of the transition matrix of a finite state Markov chain from zero-memory, scalar, nonlinear observations corrupted by additive white noise. Sawaragi et. al. [51,52] were the first to address the problem of state estimation for linear discrete-time systems with stationary interrupted observation mechanism where the statistics of the interruption process are unknown but fixed. In [51], the interruption process was assumed to be an independent binary sequence. In [52], a more general model of Markov interruption process was treated. In both these papers a Bayesian viewpoint was adopted and the objective was to present a feasible adaptive algorithm that sequentially produced an approximate minimum variance estimator. The asymptotic convergence of the a posteriori probability of the transition probabilities, given the past observations, was demonstrated through simulations. No theoretical justification was provided.

Additional comments regarding the previous work may be found in the sequel. In the next section we formulate the class of problems discussed in this thesis.

1.3. Problem Statement

In this thesis, two main classes of problems are considered. In the first one, the switching parameters are statistically modelled while in the second, multiple bounds on the values of the unknown time-varying parameters and their time-derivative(s) are considered.

The system considered is modelled by the state equation

$$x_{k+1} = Ax_k + Bw_k \tag{1.1}$$

and is observed in additive noise as

$$z_k = C_k x_k + v_k, \qquad k = 0,1,... \tag{1.2}$$

Given that x_k is n-dimensional system state vector at time instant k; z_k is m-dimensional measurement vector; $x_0 \sim N(\bar{x}_0, P_0)$; A and B are $m \times n$ and $n \times r$ constant matrices, respectively; and C_k is $m \times n$ matrix. The sequences $\{w_k\}$ and $\{v_k\}$ are r- and m-dimensional random sequences, respectively.

In the sequel based on the two approaches mentioned above, the following three cases are considered; the first two in conjunction with the Bayesian approach and the third in conjunction with the detection-estimation approach.

Case 1: In this case, $C_k = C$ for all k, where C is an $m \times n$ constant matrix. The statistics of the random sequences $\{w_k\}$ and $\{v_k\}$ are time-varying and are governed by one of S multivariate Gaussian distributions. The transitions among the distributions are Markov, with transition probabilities given by an $S \times S$ transition matrix π. Each allowable

Gaussian distribution is such that

$$E\{w_k^{(i)}\} = \mu^{(i)}, E\{v_k^{(i)}\} = \nu^{(i)}$$

$$\text{cov}(w_k^{(i)}, w_j^{(\ell)}) = Q^{(i)}\delta_{kj}\delta_{i\ell}$$

$$\text{cov}(v_k^{(i)}, v_j^{(\ell)}) = R^{(i)}\delta_{kj}\delta_{i\ell}$$

$$\text{cov}(w_k^{(i)}, v_j^{(\ell)}) = 0 \ \forall \ j,k,i,\ell; \ i,\ell = 1,2,\ldots,S$$

The superscripts refer to a specific Gaussian distribution from the set of S allowable distributions. It should therefore be noted that all the expectations given above are conditional on specific states of the Markov chain governing the transitions.

The transition probability matrix π may or may not be known. If unknown, it is assumed to belong to a finite set which also contains the true transition matrix π_t; that is, $\pi, \pi_t \in \Omega'$ where $\Omega' = \{\pi_1, \pi_2, \ldots, \pi_M\}$. Furthermore, it is assumed that the a priori distribution on the π's in Ω' is known.

Notice that if the environment (defined by a specific Markov chain state sequence) is not known, the system noise and the measurement noise will not be independent. Furthermore, v_{k+1} will not be independent of x_{k+1} since w_k and v_{k+1} are correlated; w_k influences the determination of x_{k+1}. However, when the environment is known, the dependence on environment can be removed to give independent Gaussian noise in both the system and the measurement equations.

Case 2: In this case, $C_k = \gamma_k C$ where C is an m × n constant matrix. $\{\gamma_k\}$ is a stationary Markov chain that takes on values of 0 or 1, and is called the interruption process in the sequel. The sequences $\{w_k\}$ and $\{v_k\}$ are zero-mean white Gaussian processes with constant covariances Q and R, respectively, where R is assumed to be a positive definite symmetric matrix. The random sequences w_k, v_k and γ_k are mutually independent and are also independent of initial system state x_0.

Let π be the transition probability matrix of the interruption process γ_k with elements

$$p_{ij} = P\{\gamma_k = i | \gamma_{k-1} = j\} \qquad i,j = 0,1.$$

It should be noted that the transition matrix π is characterized by two values p_{00} and p_{11}. It is assumed that the transition matrix π is an unknown constant and can take values only from a finite set $\Omega' = \{\pi_1, \pi_2, \ldots, \pi_M\}$, which also contains the true transition matrix π_t. Furthermore, the a priori distribution on the π's is assumed to be known.

Case 3: In this case, $C_k = C$ for all k, where C is an m × n constant matrix. The sequences $\{w_k\}$ and $\{v_k\}$ are independent, white, zero-mean Gaussian processes with covariances Q_k and R_k, that are positive semi-definite and positive definite, respectively. The given model is specified up to a set of unknown parameters, possibly time-varying, denoted by the vector $\theta(k)$ (at time k) of dimension q. This uncertainty may be in Q_k or R_k only, which are assumed to be continuous functions of $\theta(k)$. The unknown parameter vector $\theta(k)$ is assumed to satisfy at least one of the following conditions

$$\theta(k) \in \Omega_i, \quad i = 1,2,\ldots,N$$

where Ω_i, $i = 1,2,\ldots,N$ is a collection of compact subsets of $\mathbb{R}^q$. The sets Ω_i may be disjoint or nested. Furthermore, the sets Ω_i may specify bounds on values of $\theta(k)$ and its time-derivative(s).

The main objective in all the three cases is to obtain an estimate of the value of x_k from the past observations $Z_k = \{z_j,\ 0 \leq j \leq k\}$.

1.4. Thesis Outline

In the following chapters we study the problem of state estimation for a class of linear discrete-time dynamical systems with unknown time-varying parameters. Two different approaches to the problem are investigated. In Chapters 2 through 4 the jump parameters are assigned a probabilistic description. In Chapters 5 and 6 multiple bounds on the unknown covariances and their time derivatives are assumed to be available.

In Chapter 2 we use a finite state Markov chain model for the jump parameters. The probability transition matrix is assumed to be known. Both optimal MMSE state estimator and suboptimal approximations to it are considered in this chapter. In Chapter 3 the transition matrix is no longer assumed to be known. Attention is confined to jumps in noise statistics only. Both optimal and suboptimal Bayesian adaptive state estimators are investigated. Considerable attention is paid to the asymptotic behavior of the optimal adaptive scheme. In Chapter 4 we consider the problem of adaptive state estimation for systems under uncertain observations with unknown statistics.

In Chapters 5 and 6 a detection-estimation scheme is investigated for state estimation when information regarding the uncertainties is assumed to be available in the form of multiple bounds. In Chapter 5 we consider only time-invariant uncertainties in the noise covariances. In Chapter 6 the unknown noise covariances are allowed to be time-varying. Finally, the work is summarized in Chapter 7.

CHAPTER 2

ESTIMATION FOR SYSTEMS WITH MARKOVIAN SWITCHING PARAMETERS

2.1. Introduction

In this chapter we consider the problem of state estimation for Cases 1 and 2 of model (1.1)-(1.2) (see Section 1.3). The transition probability matrix π is assumed to be known. In order to avoid cumbersome notational problems, attention is confined mainly to state estimation in switching environments, i.e., Case 1; although the results are applicable to more general models including Case 2.

Ackerson and Fu [1] appear to have been the first to treat the problem of state estimation in switching environments. They noted that in order to evaluate the minimum mean-squared error (MMSE) estimate of the system state, the computational requirements increase exponentially with time, which makes the optimal scheme impractical. They then proposed a sub-optimal algorithm assuming that the a posteriori probability density of the state given the past observations is Gaussian. An entirely different approach was taken by H. Akashi and H. Kumamoto [2]. In their algorithm, the set of Markov chain state sequences which characterize the MMSE estimate is regarded as a population and the estimate is calculated with a relatively small number of sequences sampled at random from the population. (Other approximations given in [13,51,67] are closely related to the approach of Ackerson and Fu [1].

The suboptimal estimator of [1] is rather ad-hoc. Its main justification is that the resulting algorithm is simple. Moreover, its performance is usually good. However, simulation examples are available in [2] which indicate that this approach does not always work so well. The random sampling approach of [2] is a better approximation to the optimal estimator. However, it can be computationally demanding as a fairly large number of "samples" (of the state sequences) are needed (as in any random sampling scheme). In this chapter we investigate a detection-estimation type of approach which is intuitively and mathematically more pleasing than the approach of Ackerson and Fu [1], while also being computationally less demanding than the random sampling approach of [2].

In Section 2.2, an *MMSE estimator*, optimal in the Bayesian sense, is derived. The proposed detection-estimation approach is described in Section 2.3. Simulation results are given in Section 2.4 comparing the three approaches -- pseudo-Bayes [1], random sampling [2], and detection-estimation.

2.2. Optimal Estimator

We consider Case 1 of model (1.1)-(1.2). The transition matrix π is assumed to be known. Let $Z_k = \{z_i, 0 \le i \le k\}$ denote the collection of past observations at time k. The optimal estimate $\hat{x}(k|k)$ of the system state x_k is defined to be any function of Z_k which minimizes the mean-square error. It is well known that the MMSE estimator is given by the conditional mean

$$\hat{x}(k|k) = E\{x_k|Z_k\} \tag{2.1}$$

Define a Markov chain state sequence I(k) as

$$I(k) = \{i_0, i_1, \ldots, i_k\} \quad , \quad i_j \in S \tag{2.2}$$

where i_j represents the Markov chain state at time j; i.e., it is the index of the distribution from which w_j and v_j are sampled. Let the estimate conditional on a specific sequence of states be denoted by

$$\hat{x}_j(k|k) \triangleq E\{x_k|Z_k, I_j(k)\} \tag{2.3}$$

where $I_j(k)$ is a specific sequence from the space Ω_k of the sequences I(k). Note that the space Ω_k contains S^{k+1} elements.

On using the preceding definitions one obtains

$$\hat{x}(k|k) = \sum_{I_j(k)\in\Omega_k} \hat{x}_j(k|k)P(I_j(k)|Z_k) \tag{2.4}$$

where $P(I_j(k)|Z_k)$ denotes the conditional probability of the sequence $I_j(k)$ given the observations Z_k. The application of Bayes' rule further yields

$$P(I_j(k)|Z_k) = \frac{f(z_k|I_j(k),Z_{k-1})P(I_j(k)|Z_{k-1})}{\sum_{\ell=1}^{S^{k+1}} f(z_k|I_\ell(k),Z_{k-1})P(I_\ell(k)|Z_{k-1})} \tag{2.5}$$

where $f(z_k|I_j(k),Z_{k-1})$ is the conditional density of the k-th observation sample z_k given the earlier observations Z_{k-1} and the particular Markov chain sequence $I_j(k)$. Furthermore, we have

$$P(I_j(k)|Z_{k-1}) = P(I_\ell(k-1)|Z_{k-1})P(i_k = m|i_{k-1} = n) \tag{2.6}$$

where m, n and the sequences $I_j(k)$ and $I_\ell(k-1)$ are defined by the relations

$$I_j(k) = \{I_\ell(k-1), i_k = m\} \tag{2.7}$$

$$I_\ell(k-1) = \{i_0, i_1, \ldots, i_{k-2}, i_{k-1} = n\} \tag{2.8}$$

and $P(i_k = m|i_{k-1} = n)$ = probability that the noise sample at time k is from the m^{th} distribution given that the noise sample at time k-1 is from the n^{th} distribution.

Now $f(z_k|I_j(k), Z_{k-1})$ and $\hat{x}_j(k|k)$ can be calculated recursively by applying Kalman filtering methods to the system model (1.1)-(1.2). The initial condition x_0 has a Gaussian distribution. Given the sequence $I_j(k)$, the system (1.1)-(1.2) is now a linear system with Gaussian noise. Therefore, equations for $\hat{x}_j(k|k)$ are those for a Kalman filter [1], given by

$$\hat{x}_j(k|k) = A\hat{x}_\ell(k-1|k-1) + B\mu^{(m)} + K_j(k)[z_k - CA\hat{x}_\ell(k-1|k-1) - CB\mu^{(m)} - \nu^{(m)}] \tag{2.9}$$

$$K_j(k) = M_j(k)C^T[CM_j(k)C^T + R^{(m)}]^{-1} \tag{2.10}$$

$$M_j(k) = AP_\ell(k-1)A^T + BQ^{(m)}B^T \tag{2.11}$$

$$P_j(k) = M_j(k) - K_j(k)CM_j(k) \tag{2.12}$$

Note that $P_j(k)$ is the conditional covariance of the filtered estimate, while $M_j(k)$ is the conditional covariance of the one-step predicted estimate. From (2.7) and (2.8) it is clear that

$$f(x_{k-1}|I_j(k),Z_{k-1}) = f(x_{k-1}|I_\ell(k-1),Z_{k-1}) \sim N(\hat{x}_\ell(k-1|k-1),P_\ell(k-1)) \quad (2.13)$$

which further implies

$$f(z_k|I_j(k),Z_{k-1}) \sim N\{C(A\hat{x}_\ell(k-1|k-1) + B\mu^{(m)}) + \nu^{(m)},CM_j(k)C^T+R^{(m)}\} \quad (2.14)$$

where $N(\mu,D)$ is the standard notation for normal density with mean μ and covariance matrix D. The expressions (2.3)-(2.14) yield the recursive optimal MMSE estimate $\hat{x}(k|k)$.

It is obvious from Equation (2.4) that the optimal estimator requires an exponentially increasing memory. Specifically, at the k^{th} step, S^{k+1} Kalman filters are run to compute $\hat{x}_j(k|k)$. Moreover, calculation of $P(I_j(k)|Z_k)$ requires knowledge of S^k probabilities $P(I_\ell(k-1)|Z_{k-1})$. So the procedure given in this section, while optimal, is not very practical. To circumvent this difficulty, various approximations to the optimal MMSE estimator have been proposed in the literature [1,2,13,67]. The sub-optimal estimator of [1], called the pseudo-Bayes approximation, is summarized in Appendix A. The random sampling approach [2] is discussed in Appendix B. In the next section, we discuss a new approximation to the optimal estimator.

2.3. Detection-Estimation Approach

In this section, we discuss a new approximation to the optimal state estimator in switching environments. Recall that in the optimal solution, at stage k, we need S^{k+1} Kalman filters matched to all possible Markov chain state sequences $I_j(k)$, $j = 1,2,\ldots,S^{k+1}$. The objective now is to

control the number of Kalman filters to be a finite, and below a maximum allowable, number. Instead of carrying the probabilities $P(I_j(k)|Z_k)$ in (2.4) for all j, we disregard some of the "unlikely" sequences, i.e., we do not process the Kalman filters matched to these unlikely sequences.

Consider the conditional probability

$$P(I_j(k)|Z_k) = f(Z_k|I_j(k))P(I_j(k))/f(Z_k), \tag{2.15}$$

in which the denominator is common for all values of j, so that only the numerator need be considered. Given $I_j(k)$, the conditional density $f(Z_k|I_j(k))$ is a Gaussian density function and can easily be computed recursively using Kalman filtering methods [1]. Let M denote the maximum number of state sequences (or equivalently, Kalman filters) to be processed at any stage k. Suppose that, at stage k-1, we have selected N_{k-1} state sequences ($N_{k-1} \leq M$). At stage k, consider all possible "extensions" of these N_{k-1} state sequences. By an extension I(k) of a state sequence $I_j(k-1)$ we mean

$$I(k) = \{I_j(k-1), i(k) = m\}, \qquad m = 1,2,\ldots,S,$$

so that S extensions of $I_j(k-1)$ are possible. Out of these $N_{k-1} \times S$ state sequences, we retain N'_{k-1} sequences which satisfy

$$\|z_k - \hat{z}_j(k|k-1)\|^2_{P_j^{-1}(k|k-1;z)} \leq \beta^{\ddagger} \tag{2.16}$$

where

$\hat{z}_j(k|k-1)$ = prediction of z_k based on the observation Z_{k-1} assuming an extension (indexed by j) of one of the N_{k-1} sequences to be true

$P_j(k|k-1;z)$ = corresponding prediction error covariance

and β = a positive number.

In other words, let $I_j(k)$ be an "extended" sequence where $j = 1,2,\ldots,(N_{k-1} \times S)$. Then

$$\hat{z}_j(k|k-1) = E\{z_k|Z_{k-1}, I_j(k)\}$$

and

$$P_j(k|k-1;z) = E\{[z_k - \hat{z}_j(k|k-1)][z_k - \hat{z}_j(k|k-1)]^T|I_j(k)\}$$

Note that if the assumed state sequence $I_j(k)$ is indeed true, then the left-hand side of (2.16) is a chi-square random variable with m degrees of freedom where m is the dimension of z_k. Hence β can be selected to correspond to a fixed probability of rejecting the correct state sequence (the probability of a "miss"). Thus the criterion (2.16) results in the selection of N'_{k-1} "most likely" state sequences. Note that the test (2.16) represents the requirement that a candidate state sequence should lie in the ellipsoid of a given probability concentration.

$^{\ddagger}\|z_k-\hat{z}_j(k|k-1)\|^2_{P_j^{-1}(k|k-1;z)} \triangleq (z_k-\hat{z}_j(k|k-1))^T P_j^{-1}(k|k-1;z)(z_k-\hat{z}_j(k|k-1))$

Now if $N'_{k-1} \leq M$, we may carry along all these state sequences and set $N_k = N'_{k-1}$. But if $M < N'_{k-1}$, then M "most likely" candidate sequences are selected according to the following procedure. Arrange $f(Z_k|I_j(k))P(I_j(k))$, $j = 1,2,\ldots,N'_{k-1}$, in decreasing order of magnitude and select the state sequences $I_j(k)$ corresponding to the first M probabilities $f(Z_k|I_j(k))P(I_j(k))$ as the M candidate sequences to be carried forward, resulting in $N_k = M$.

Equation (2.4) is now modified as

$$\hat{x}(k|k) = \sum_{j=1}^{N_k} \hat{x}_j(k|k)P(I_j(k)|Z_k) \tag{2.17}$$

where

$$P(I_j(k)|Z_k) = \frac{f(Z_k|I_j(k))P(I_j(k))}{\sum_{j=1}^{N_k} f(Z_k|I_j(k))P(I_j(k))} \tag{2.18}$$

Notice that the subscript j now indexes the selected N_k state sequences.

A further refinement of this procedure may be required. It is possible that two state sequences which differ in "early" states (and have the same "recent" states), may merge into each other in that the predictions $\hat{z}_j(k|k-1)$ based on either sequence are almost the "same". Under such a situation, there is no point in carrying along both these sequences. We shall regard two state sequences $I_j(k)$ and $I_\ell(k)$ as distinguishable only if the Bhattacharyya distance (B-distance) [29] between the conditional probability density functions $f(z_k|Z_{k-1},I_j(k))$ and $f(z_k|Z_{k-1},I_\ell(k))$ is more

than some real number α. Out of the two sequences $I_j(k)$ and $I_\ell(k)$ for which the B-distance $d(I_j(k), I_\ell(k)) \leq \alpha$, we choose to keep that sequence for which the corresponding prediction error, in the sense of (2.16), is smaller. Then, in (2.18), the probability $P(I_j(k))$ is appropriately modified by "absorbing" the probability of the dropped sequence.

The motivation for calling the proposed scheme a detection-estimation approach should be apparent by now. A fixed number of most likely state sequences (detection) characterize the state estimate (estimation).

Selection of M

Recall that M is the maximum allowable number of state sequences to be carried forward at any stage k. Consider the problem where the probability of no jump from a given state to any other Markov chain state equals p and is the same for all the states. Then the probability of a jump (to any other state) equals q = 1-p. Furthermore, let

N = "block" length over which processing is to be done

= k+1 at stage k

Then, average number of jumps in the block = Nq, with standard deviation = $(Nqp)^{\frac{1}{2}}$. Therefore, number of jumps $\leq Nq + 3(Nqp)^{\frac{1}{2}}$ with probability > 0.99, if N is large. Now, let

J = number of jumps in a given block length

L = number of different state sequences of length N

Then, with probability > 0.99, we have

$$L \leq S\left[\sum_{k=0}^{J} \binom{N-1}{k}(S-1)^k\right] \tag{2.19}$$

where S is the number of states in the Markov chain. The term inside the square brackets in (2.19) gives the maximum number of different sequences with the "first" state fixed and with number of jumps $\leq J$. As examples consider the following cases:

Example 2.1: $S = 2$, $N = 10$, $q = 0.1$, $p = 0.9$

Then $J \simeq 4$

Therefore, $L \leq 512$ as opposed to 2^{10} sequences in the optimal estimator.

Example 2.2: $S = 2$, $N = 10$, $q = 0.01$, $p = 0.99$

Then $J \simeq 1$ and $L \leq 20$ as opposed to 2^{10} sequences.

Example 2.3: $S = 2$, $N = 10$, $q = 0.99$, $p = 0.01$

Define $\bar{J}$ = number of "no" jumps in block length N

Also, let $q' = p$ and $p' = q$, and use q' and p' in (2.19) instead of q and p. Then, $\bar{J} \simeq 1$ and $L \leq 20$ as in Example 2.2.

Notice that the given procedure is useful only in the extreme cases of very high or very low jump probabilities as in Examples 2.2 and 2.3. Otherwise the choice of M is usually dictated by the computation and storage capabilities of the processor.

2.4. Numerical Example

In this section, a simulation example is presented to compare the three (suboptimal) approaches to state estimation in switching environments - pseudo-Bayes (that of [1]), random sampling, and detection-estimation. The example has been taken from [2].

Consider a scalar dynamical system described by the following equations:

$$x_{k+1} = 1.04\, x_k + w_k$$

$$z_k = x_k + K(i_k)v_k + a(i_k), \quad k = 0,1,2,\ldots$$

$$S = 2, \text{ i.e., } i_k \in \{1,2\}.$$

Initial conditions are: $x_0 \sim N(30,400)$. In the actual system we use $x_0 = 1$ for simulation purposes. $\{w_k\}$ and $\{v_k\}$ are zero-mean white Gaussian sequences with covariances $Q = 0.1$ and $R = 1.0$, respectively. The transition probabilities are given by

$$P(i_{k+1} = 1 | i_k = 1) = 0.85$$

$$P(i_{k+1} = 2 | i_k = 1) = 0.15$$

$$P(i_{k+1} = 2 | i_k = 2) = 0.3$$

$$P(i_{k+1} = 1 | i_k = 2) = 0.7$$

Furthermore, $P(i_0 = 1) = P(i_0 = 2) = 0.5$. The three suboptimal estimators were simulated and their performances are compared in Figs. 2.1 and 2.2. The performances were averaged over 50 Monte Carlo runs (same as in [2]). For Fig. 2.1, $K(1) = 40$, $K(2) = 1$, $a(1) = a(2) = 0$. For Fig. 2.2, $K(1) = 40$, $K(2) = 1$, $a(1) = 50$, $a(2) = 0$. For both these figures, the random sampling approach was applied using 100 samples of state sequences (as in [2]) whereas the detection-estimation approach was applied using $M = 20$. That is, 100 filters were needed for the random-sampling approach whereas only 20 were used for the detection-estimation approach. Furthermore, for the detection-estimation scheme, $\beta = 6.63$ in (2.16) which corresponds to a "miss" probability of 0.01. The B-distance feature was not incorporated.

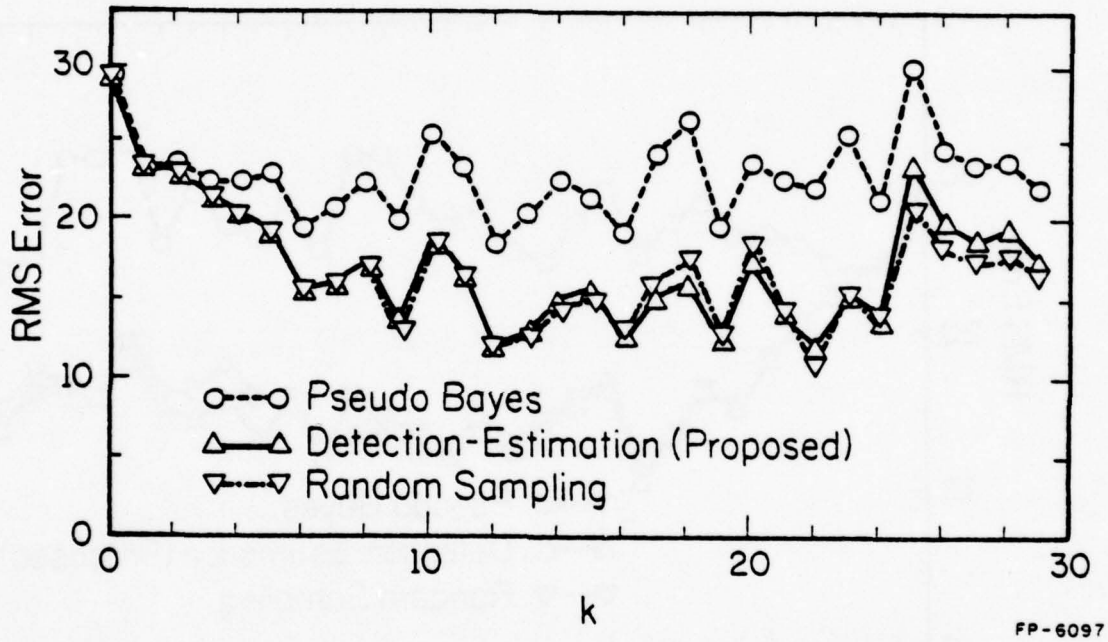

Figure 2.1. Comparison of the RMS errors in state estimates due to the suboptimal estimators, for K(1) = 40, K(2) = 1, a(1) = a(2) = 0.

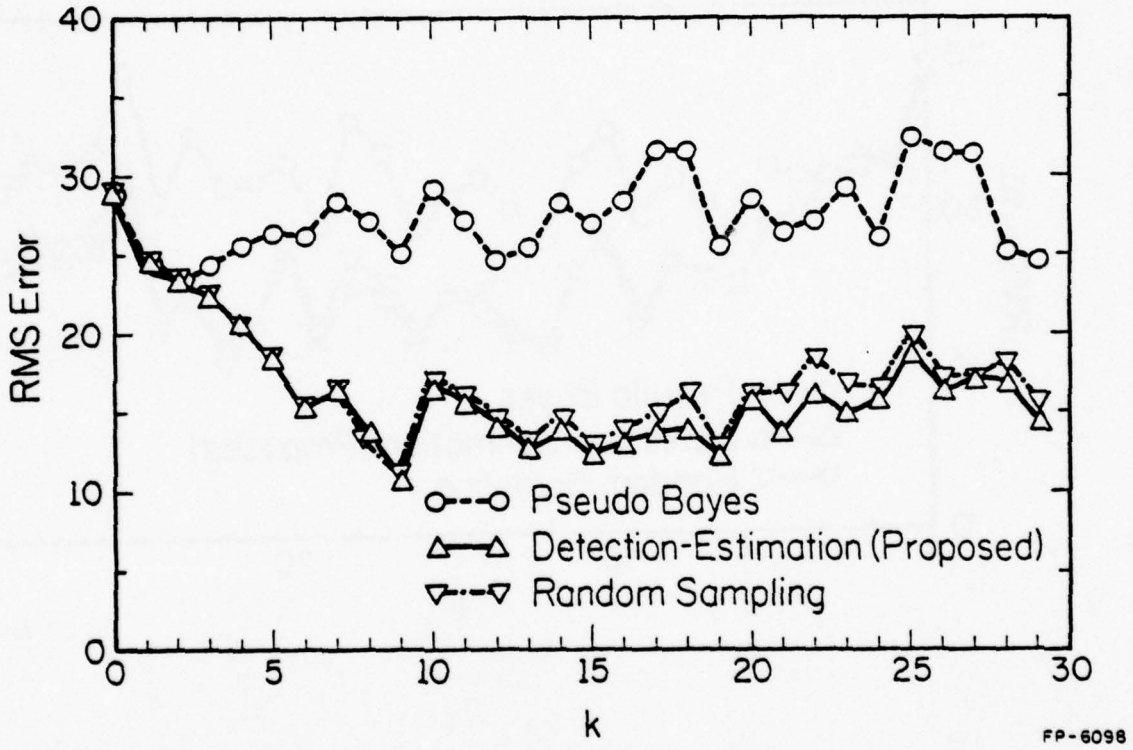

Figure 2.2. Comparison of the RMS errors in state estimates due to the suboptimal estimators, for K(1) = 40, K(2) = 1, a(1) = 50, a(2) = 0.

From both these figures, it is clear that both the random-sampling approach and the detection-estimation approach, are superior to the pseudo-Bayes approximation. The performances of the random-sampling and the detection-estimation approaches are quite close to each other. However, from a computational viewpoint, the detection-estimation approach is preferable. It should be noted that a random sampling scheme has to use a fairly large number of samples in order for the results to have some statistical significance.

The results of more extensive simulations are shown in Table 2.1 for parameter values K(1) = 40, K(2) = 0, a(1) = 50, a(2) = 0. The combined ensemble- and time-averages of r.m.s. errors due to the three suboptimal estimators are compared. The r.m.s. error in the state estimate was first averaged over 50 Monte Carlo runs, then it was averaged over 30 stages. Let D_t denote the combined ensemble- and time-average of r.m.s. error in the state estimate, and let $D_e(k)$ denote the ensemble-average (over 50 Monte Carlo runs) of r.m.s. error at time k. Then we have

$$D_t = \frac{1}{30} \sum_{k=0}^{29} D_e(k).$$

Table 2.1 shows the average r.m.s. error D_t. For the detection-estimation approach, the maximum allowed number M of the Kalman filters was varied from 5 to 20 in steps of 5. For the random sampling approach, the number of sequences sampled were varied from 10 to 100. It is clear from the table that, for a given degree of accuracy, the detection-estimation approach is computationally superior to the random sampling approach.

TABLE 2.1

Average r.m.s. error D_t for the three suboptimal algorithms.

Scheme	Average r.m.s. Error D_t
Pseudo-Bayes	28.5814
Detection-Estimation	
with M = 20	17.2223
15	17.2217
10	17.1503
5	17.7486
Random Sampling with	
Number of Sequences	
Sampled = 100	17.2256
60	17.5082
40	17.6131
20	18.4488
10	19.7364

2.5. Discussion

A combined detection-estimation type of approach has been proposed for state estimation in linear dynamical systems operating in switching environments. Simulation results were presented which indicated the advantages of the proposed approach over the pseudo-Bayes and the random sampling approaches.

We note that the proposed approach can easily be generalized for state-estimation in a larger class of linear discrete-time systems, namely, linear systems with Markovian, jump parameters where jumps may occur in noise statistics or in matrices A, B and C. Also, the matrices A, B and C may be taken to be time-varying.

CHAPTER 3

ADAPTIVE ESTIMATION IN SWITCHING ENVIRONMENTS

3.1. Introduction

In the previous chapter we considered the problem of state estimation in switching environments under the assumption that the transition probability matrix was known. This restriction is removed in this chapter. We assume that the transition matrix is unknown and can take values only from a finite set, which also contains the true value of the transition matrix. Previous work on state estimation in switching environments has not investigated the problem of unknown transition statistics considered above.

A Bayesian approach is adopted for the solution of the problem. The objectives are to present a feasible adaptive algorithm that sequentially produces an approximate MMSE estimator, and to investigate the asymptotic behavior of the Bayes optimal adaptive estimation scheme. Specifically, conditions are investigated under which the constrained maximum likelihood estimate of the transition matrix converges in probability to the correct value of the transition matrix. Then convergence of the a posteriori probability of the transition matrix given the past observations is obtained. Some results on the convergence of the performance of the optimal adaptive scheme are also presented.

In Section 3.2 an MMSE estimator, optimal in the Bayesian sense, is derived. Suboptimal estimators are discussed in Section 3.3. In Section 3.4 conditions for the convergence of the estimates (maximum likelihood and Bayes) of the transition matrix are discussed. Convergence in performance is given in Section 3.5. An example to illustrate the convergence results is given in Section 3.6.

Note: In the following sections, the distinction between a random vector and its particular realization (value) will not (always) be made in the notation, rather it will be left to context. Also, unless otherwise indicated, the probability measure used throughout is $P_{\pi_t}^{Z_n}$ which is a probability measure induced on the range space of the random sequence $Z_n \triangleq \{z_0,\ldots,z_n\}$ parameterized by the correct value π_t of the transition matrix π.

3.2. Optimal Estimator

We consider Case 1 of model (1.1)-(1.2). The transition matrix π is unknown and can take values only from a finite set $\{\pi_1,\pi_2,\ldots,\pi_M\}$. Let $P(\pi_q) \triangleq P\{\pi = \pi_q\}$ denote the a priori probability of the q^{th} transition matrix. The problem is to obtain an estimate of the state x_k after each observation and to investigate its asymptotic behavior.

Let $Z_k = \{z_i,\ 0 \le i \le k\}$ denote the collection of past observations at time k. The optimal estimate $\hat{x}(k|k)$ of the system state x_k is defined to be any function of Z_k which minimizes the mean-square error. It is well known that the MMSE estimator is given by the conditional mean

$$\hat{x}(k|k) = E\{x_k|Z_k\} \tag{3.1}$$

Let $\hat{x}(k|k;\pi_q) = E\{x_k|Z_k,\pi_q\}$ denote the conditionally optimal estimate of x_k given π_q and Z_k. Let $P(\pi_q|Z_k)$ denote the conditional probability of π_q given data Z_k. Then we have

$$\hat{x}(k|k) = \sum_{q=1}^{M} \hat{x}(k|k;\pi_q)P(\pi_q|Z_k) \tag{3.2}$$

The use of Bayes' rule results in

$$P(\pi_q|Z_k) = \frac{f(Z_k|\pi_q)p(\pi_q)}{\sum_{q=1}^{M} f(Z_k|\pi_q)P(\pi_q)} \tag{3.3}$$

where $f(Z_k|\pi_q)$ is the conditional density function of Z_k given $\pi = \pi_q$. It should be noted that $\hat{x}(k|k;\pi_q)$ has already been derived in Section 2.2, and parts of the derivation are repeated here due to changes in notation caused by π_q. Let the Markov chain state sequence be denoted by I(k) and let $I_j(k)$ denote a specific such sequence as in Equation (2.7). Similarly, the estimate conditional on a specific sequence of states is denoted by

$$\begin{aligned}\hat{x}_j(k|k;\pi_q) &\triangleq E\{x_k|\pi_q,Z_k,I_j(k)\} \\ &= E\{x_k|Z_k,I_j(k)\} = \hat{x}_j(k|k)\end{aligned} \tag{3.4}$$

It is seen that the estimate does not depend on the transition matrix π_q. On using the preceding definitions one obtains

$$\hat{x}(k|k;\pi_q) = \sum_{I_j(k)\in\Omega_k} \hat{x}_j(k|k)P(I_j(k)|Z_k,\pi_q) \tag{3.5}$$

where Ω_k is the space of all sequences I(k). The application of Bayes' rule further yields

$$P(I_j(k)|Z_k,\pi_q) = \frac{f(z_k|I_j(k),Z_{k-1})P(I_j(k)|Z_{k-1},\pi_q)}{\sum_{\ell=1}^{S^{k+1}} f(z_k|I_\ell(k),Z_{k-1})P(I_\ell(k)|Z_{k-1},\pi_q)} \tag{3.6}$$

where we write $f(z_k|I_j(k),Z_{k-1})$ for $f(z_k|I_j(k),Z_{k-1},\pi_q)$ as the latter does not depend on π_q. Furthermore, we have

$$P(I_j(k)|Z_{k-1},\pi_q) = P(I_\ell(k-1)|Z_{k-1},\pi_q)P(i_k=m|i_{k-1}=n,\pi_q) \tag{3.7}$$

where $I_j(k), I_\ell(k-1)$, m and n are as defined in (2.7)-(2.8) and $P(i_k=m|i_{k-1}=n,\pi_q)$ = probability that the noise sample at time k is from the m^{th} distribution given that the noise sample at time k-1 is from the n^{th} distribution and given that $\pi = \pi_q$.

Now $f(z_k|I_j(k),Z_{k-1})$ and $\hat{x}_j(k|k)$ can be calculated recursively by applying Kalman filtering methods to the system model (1.1)-(1.2), as in Equations (2.9)-(2.14).

It remains to calculate $P(\pi_q|Z_k)$ to complete calculation of $\hat{x}(k|k)$. We have

$$f(Z_k|\pi_q) = \prod_{n=0}^{k} f(z_n|\pi_q,Z_{n-1}) \tag{3.8}$$

where

$$f(z_0|\pi_q,Z_{-1}) \triangleq f(z_0|\pi_q)$$

and

$$f(z_k|Z_{k-1},\pi_q) = \sum_{I_j(k)\in\Omega_k} f(z_k|I_j(k),Z_{k-1})P(I_j(k)|\pi_q,Z_{k-1}) \qquad (3.9)$$

Expressions for quantities on the right side of Equation (3.9) have already been obtained in recursive form. Hence $f(Z_k|\pi_q)$ can be calculated recursively for each π_q. Then (3.3) may be used to compute $P(\pi_q|Z_k)$ for each π_q.

As pointed out earlier in Section 2.2 and as is obvious from Equation (3.5), the optimal estimator requires exponentially increasing storage (memory) and computation capabilities. In the next section we discuss approximations to the optimal estimator in order to reduce the large storage and computation requirements.

3.3. Approximations to the Optimal Estimator

In this section we briefly discuss several approximations to the optimal estimator in order to reduce the large storage and computational requirements of the optimal estimator. These approximations are based on those outlined in Chapter 2.

We want to develop approximations to the optimal estimator given by (3.2). Now any of the three approximations (random sampling, pseudo-Bayes, detection-estimation) discussed in Chapter 2 may be used to approximate $\hat{x}(k|k;\pi_q)$, given by Equation (3.5), for a fixed value of π. So the procedure has to be repeated M times to get approximations to $\hat{x}(k|k;\pi_q)$ for each admissible π_q, $q = 1,2,\ldots,M$. It remains to specify corresponding approximations to $P(\pi_q|Z_k)$ given by (3.3), in order to complete the calculation of $\hat{x}(k|k)$ given by (3.2). These approximations are discussed below:

Random Sampling: In this method $f(Z_k|\pi_q)$ in (3.3) is approximated by the denominator of Equation (B.6) (see also Equation (B.8) and Property P3 in Appendix B) for a fixed value π_q. The procedure may then be repeated for each admissible π_q, q = 1,2,...,M. This computation is a by-product of the calculation performed to obtain $\hat{x}(k|k;\pi_q)$.

Pseudo-Bayes: In this case we use Equations (3.10) (given below) and (3.8) to obtain an approximation to $f(Z_k|\pi_q)$ for a fixed π_q. Equation (3.10) is a pseudo-Bayes approximation (see Appendix A) to (3.9)

$$f(z_k|Z_{k-1},\pi_q) = \sum_{\ell=1}^{S} f(z_k|i_k=\ell,Z_{k-1},\pi_q)P(i_k=\ell|Z_{k-1},\pi_q) \qquad (3.10)$$

The pseudo-Bayes approximations to the quantities on the right-hand side of (3.10), for a fixed π_q, are given in Appendix A. Again these computations are by-products of the calculations performed to obtain $\hat{x}(k|k;\pi_q)$.

Detection-Estimation: In this approach the denominator of Equation (2.18) is taken to be an approximation to $f(Z_k|\pi_q)$ for a fixed π_q. Once again this computation is a by-product of the calculation for obtaining $\hat{x}(k|k;\pi_q)$.

Other approximations based on those given in [27] and [67] may also be used. These are closely related to the pseudo-Bayes approximation.

3.4. Convergence of Transitions Matrix Estimates

In this section, the conditions under which the a posteriori probability of the true value π_t of the transition matrix converges (weakly) to 1 are investigated. Note that the solution given in Section 3.3 is optimum in the Bayes sense, at every step k, and the optimality,

therefore, is independent of convergence. It is, however, of interest to know conditions under which the solution converges to one that uses the correct transition matrix value, since it makes the asymptotic solution independent of the a priori probabilities $P(\pi_q)$.

The results are developed via weak consistency of maximum likelihood (ML) estimate of π. The constrained maximum likelihood estimate of π_t given observation Z_n is defined as $\hat{\pi}(Z_n)$ which satisfies the equation

$$f(Z_n|\hat{\pi}(Z_n)) = \max_{\pi \in \Omega'} f(Z_n|\pi), \; \Omega' = \{\pi_1,\ldots,\pi_M\}.$$

A sequence of estimates $\{\hat{\pi}(Z_n),\ 1 \leq n \leq \infty\}$ is said to be weakly consistent (or consistent in probability), if for any positive δ, ϵ, there exists an $N(\epsilon,\delta) < \infty$ such that for $n > N(\epsilon,\delta)$ $P\{d(\hat{\pi}(Z_n),\pi_t) > \delta\} < \epsilon$ where d is an appropriate distance function and π_t is the correct value of π.

The following conditions are imposed:

(H1) The state transition matrix A in the system model (1.1)-(1.2) has all its eigenvalues inside the unit circle.

(H2) The noise covariance matrices defined in Section 1.3 are positive definite. That is, $Q^{(i)} > 0$, $R^{(i)} > 0$; $i = 1,2,\ldots,S$, where $B > 0$ implies that B is a positive-definite matrix. A consequence of this is that there exist upper bounds QU and RU, and lower bounds QL and RL, on $Q^{(i)}$ and $R^{(i)}$, respectively, such that

$$QU-Q^{(i)} \geq 0, \qquad RU-R^{(i)} \geq 0, \qquad i = 1,2,\ldots,S$$

$$QL-Q^{(i)} \leq 0, \qquad RL-R^{(i)} \leq 0$$

$$QL,RL > 0.$$

Here $B \geq 0$ implies that B is a positive-semidefinite matrix and $B \leq 0$ implies that B is negative-semidefinite.

(H3) The dynamical system (1.1)-(1.2) is completely observable and completely controllable [28] when the driving noise w_k and the measurement noise v_k are independent, white Gaussian sequences with "constant" covariances QL and RL, respectively. Hence, the same holds when w_k and v_k have covariances QU and RU, respectively, for all k.

(H4) The homogeneous Markov chain governed by the transition matrix π, $\pi \in \{\pi_1, \pi_2, \ldots, \pi_M\}$, is irreducible and the states of the chain are aperiodic [19].

(H5) Suppose that the system (1.1)-(1.2) begins at initial time $k = k_o$. Let $I_j(n,k)$ represent a specific sequence from the set of Markov chain state sequences $I(n,k) \triangleq \{i_k, i_{k+1}, \ldots, i_n\}$, $n \geq k$, $j = 1,2,\ldots,S^{n-k+1}$. Let $Z_{n,k_o} = \{z_k, k_o \leq k \leq n\}$ and the following is assumed:

$$\lim_{k_o \to -\infty} P(I_j(n,k)|\pi_q, Z_{n,k_o}) = P(I_j(n,k)|\pi_q, Z_{n,-\infty}) \quad \text{a.e. } P^Z_{\pi_t},$$

where $n,k > -\infty$ are fixed, and $P^Z_{\pi_t}$ is a probability measure induced on the range space of the random sequence $\{\ldots, z_k, \ldots, z_n\}$ parameterized by the correct value π_t of the transition matrix π. (Note that here we are using the same notation for a random variable and its values.)

(H6) The inequality

$$\lim_{n \to \infty} f(z_n|Z_{n-1},\pi_q) \neq \lim_{n \to \infty} f(z_n|Z_{n-1},\pi_t)$$

holds with nonzero probability, for all $\pi_q \in \{\pi_1,\ldots,\pi_M\}$, $\pi_q \neq \pi_t$, where $Z_n = \{z_k,\ 0 \leq k \leq n\}$.

Remark 3.1: A consequence of condition (H4) is that $\lim_{n \to \infty} \pi^n = \pi_\infty$ exists and is unique, i.e., there exists a unique stationary distribution for the Markov chain corresponding to π. Then $\lim_{n \to \infty} p_{jk}^{(n)} = a_k$ independent of j, where $p_{jk}^{(n)}$ is the transition probability from state j to state k in n steps [19]. Here a_k is the steady-state probability of the Markov chain being in state k. Further, the distribution of the entire (Markov chain) process, with arbitrary initial distribution, converges to the distribution of the stationary process with transition matrix π_∞ and initial distribution given by a_k's (see Proposition 7.12 in Breiman [10]).

The following theorem states the weak consistency of the constrained ML estimate of π.

Theorem 3.1: Suppose that conditions (H1)-(H6) hold. Suppose that the noise sequences $\{w_k\}$ and $\{v_k\}$ in model (1.1)-(1.2) have zero means. Then the constrained maximum likelihood estimate $\hat{\pi}(Z_n)$ of π converges in probability to π_t as $n \to \infty$.

Tse and Anton [63] have given sufficient conditions for the weak consistency of ML estimate to hold. We shall show that these conditions are satisfied for system model (1.1)-(1.2) under the conditions (H1)-(H6). Several preliminary results are needed first.

Suppose that a linear discrete-time system is described by

$$x_{k+1} = Ax_k + Bw_k$$
$$z_k = Cx_k + v_k$$
$$w_k \sim N[0,Q(k)],\ v_k \sim N[0,R(k)],\ \bar{x}_0 \sim N[\hat{x}_0,P(0)] \tag{3.11}$$

whereas its model contains noise covariances modeling errors, namely, w_k and v_k are assumed to be

$$\bar{w}_k \sim N[0,Q_c(k)], \bar{v}_k \sim N[0,R_c(k)], \bar{x}_0 \sim N[\hat{x}_0,P_c(0)], \tag{3.12}$$

where $R(k), R_c(k) > 0$; $Q_c(k) \geq 0$; $P(0) \geq 0$, $P_c(0) \geq 0$.

The use of the model (3.12) results in an MMSE filter given by

$$\hat{x}(k+1|k) = A\hat{x}(k|k)$$
$$\hat{x}(k|k) = [I - K_c(k)C]\hat{x}(k|k-1) + K_c(k)z_k \tag{3.13}$$

where

$$\hat{x}(k+1|k) \triangleq E_c\{x_{k+1}|Z_k\}^{\dagger}$$
$$\hat{x}(k|k) \triangleq E_c\{x_k|Z_k\}$$
$$Z_k \triangleq \{z_j,\ j = 0,1,\ldots,k\}$$
$$K_c(k) = P_c(k|k-1)C^T[C\ P_c(k|k-1)C^T + R_c(k)]^{-1} \tag{3.14}$$

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†Subscript c implies that the expectation is taken assuming model (3.12) to be true.

and

$$P_c(k+1|k) = A\,P_c(k|k)A^T + BQ_c(k+1)B^T$$

$$P_c(k|k) = [I - K_c(k)C]P_c(k|k-1)$$

$$= [I - K_c(k)C]P_c(k|k-1)[I - K_c(k)C]^T + K_c(k)R_c(k)K_c^T(k)$$

$$P_c(0|0) = P_c(0) \tag{3.15}$$

Now the computed matrix $P_c(k|k)$ is not the estimation error covariance matrix, since the filter model differs from the real model. Neither is this filter the MMSE filter for the actual system (3.11). The actual estimation errors are

$$\tilde{x}(k+1|k) \triangleq x_{k+1} - \hat{x}(k+1|k) = A\tilde{x}(k|k) + w_k$$

$$\tilde{x}(k|k) \triangleq x_k - \hat{x}(k|k) = [I-K_c(k)C]\tilde{x}(k|k-1) - K_c(k)v_k \tag{3.16}$$

with covariances defined by

$$P(k|k) \triangleq E\{\tilde{x}(k|k)\tilde{x}^T(k|k)\}$$

$$P(k+1|k) \triangleq E\{\tilde{x}(k+1|k)\tilde{x}^T(k+1|k)\} \tag{3.17}$$

and satisfying the relations

$$P(k+1|k) = AP(k|k)A^T + BQ(k+1)B^T$$

$$P(k|k) = [I-K_c(k)C]P(k|k-1)[I-K_c(k)C]^T + K_c(k)R(k)K_c^T(k) \tag{3.18}$$

Lemma 3.1: Let $\hat{z}(k|k-1;j)$ denote the prediction of z_k based on the observation sequence Z_{k-1}, assuming that Z_{k-1} results from model (1.1)-(1.2) with Markov chain state sequence $I_j(k)$ in effect, where $I_j(k) = \{i_n\}_{n=0}^{k}$, $j = 1,2,\ldots,s^{k+1}$, and i_n represents state at time n. Let $I_U(k)$ represent a state sequence in which the states correspond to

driving noise covariance QU and measurement noise covariance RU, for all time. Similarly, let $I_L(k)$ represent a state sequence in which the states correspond to noise covariances QL and RL for all time. Then

$$P_z(k|k-1;L|L) \leq P_z(k|k-1;j|L) \leq P_z(k|k-1;j|j), \ \forall\ j \text{ and for } k \geq 0,$$

where

$$P_z(k|k-1;L|L) \triangleq E\{(z_k-\hat{z}(k|k-1;L))(z_k-\hat{z}(k|k-1;L))^T|I_L(k)\}^{\dagger}$$

$$P_z(k|k-1;j|L) \triangleq E\{(z_k-\hat{z}(k|k-1;j))(z_k-\hat{z}(k|k-1;j))^T|I_L(k)\}$$

and

$$P_z(k|k-1;j|j) \triangleq E\{(z_k-\hat{z}(k|k-1;j))(z_k-\hat{z}(k|k-1;j))^T|I_j(k)\}$$

Proof: The desired results will be proved using models (3.11) and (3.12). From (3.15) and (3.18), let

$$T(k|k) \triangleq P(k|k) - P_c(k|k)$$

$$T(k+1|k) \triangleq P(k+1|k) - P_c(k+1|k).$$

Then

$$T(k+1|k) = AT(k|k)A^T + B[Q(k+1)-Q_c(k+1)]B^T$$

$$T(k|k) = [I-K_c(k)C]T(k|k-1)[I-K_c(k)C]^T + K_c(k)[R(k) - R_c(k)]K_c^T(k) \qquad (3.19)$$

Now suppose $Q(k) \leq Q_c(k)$ and $R(k) \leq R_c(k)$ $\forall k$. Then if $T(k|k-1) \leq 0$, it is evident from (3.19) that $T(k|k) \leq 0$ and $T(k+k|k) \leq 0$. Therefore, by induction, we have: If $P(0) \leq P_c(0)$, $Q(k) \leq Q_c(k)$ and $R(k) \leq R_c(k)$ $\forall k$, then $P(k|k) \leq P_c(k|k)$ and $P(k+1|k) \leq P_c(k+1|k)$ $\forall k$. Similarly, if $P(0) \geq P_c(0)$, $Q(k) \geq Q_c(k)$ and $R(k) \geq R_c(k)$ $\forall k$, then $P(k|k) \geq P_c(k|k)$ and $P(k+1|k) \geq P_c(k+1|k)$ $\forall k$. Further, similar results hold if we compare

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$^{\dagger}E\{(z_k-\hat{z}(k|k-1;L))(z_k-\hat{z}(k|k-1;L))^T|I_L(k)\}$ denotes expectation of $(z_k-\hat{z}(k|k-1;L))(—)^T$ assuming Markov chain state sequence $I_L(k)$ to be in effect in model (1.1)-(1.2).

actual estimation error covariance matrices of two different "mismatched" filters. This proves the following:

$$P_x(k|k-1;j|L) \leq P_x(k|k-1;j|j), \ \forall\, j, k$$

where

$$P_x(k|k-1;j|L) \triangleq E\{(x_k-\hat{x}_j(k|k-1))(x_k-\hat{x}_j(k|k-1))^T|\, I_L(k)\} \text{ etc.}$$

(Recall condition (H2) in Section 3.4.) Now since

$$P_z(k|k-1;j|L) = CP_x(k|k-1;j|L)C^T + RL, \text{ etc.}$$

we have the following:

$$P_z(k|k-1;j|L) \leq P_z(k|k-1;j|j), \ \forall\, j, k\,.$$

Since a Kalman filter matched to the true model statistics is a MMSE estimator, we have

$$P_x(k|k-1;L|L) \leq P_x(k|k-1;j|L) \quad j = 1,2,\ldots,S^{k+1};\ k = 0,1,\ldots \text{ Q.E.D.}$$

Note: Proof of $P_x(k|k-1;j|L) \leq P_x(k|k-1;j|j)$ follows the proof of Theorem 7.6 in [28].

Lemma 3.2: Let condition (H3) in Section 3.4 be satisfied. Then there exist positive real numbers δ_1 and δ_2 such that

$$0 < \delta_1 I \leq P_x(k|k;j|j) \leq \delta_2 I, \ \forall\, k \geq N > 0 \text{ and } \forall\, j\,.$$

Proof: Since, by condition (H3), system model (1.1)-(1.2) is completely controllable and completely observable for the driving noise w_k and the measurement noise v_k having covariances QL and RL, respectively, for all time, it follows from Lemma 7.2 in [28] (see also [25] and [14]) that there exists $\delta_1 > 0$ and integer $N > 0$ such that

$$P_x(k|k;L|L) \geq \delta_1 I > 0, \ \forall\, k \geq N. \tag{3.20}$$

Now consider a suboptimal estimator which uses a fixed estimate $\bar{x}_0$ for all time and for all possible Markov chain state sequences; that is, we

replace $\hat{x}_j(k|k)$ by $\bar{x}_0$ for all k and j. Since Kalman filter is the optimal MMSE estimator, we have

$$P_x(k|k;j|j) \leq E\{(x_k-\bar{x}_0)(x_k-\bar{x}_0)^T|I_j(k)\} \quad \forall j \text{ and } k \tag{3.21}$$

Now from conditions (H1) and (H2) there exists a positive number $M < \infty$ such that

$$\| E\{(x_k-\bar{x}_0)(x_k-\bar{x}_0)^T|I_j(k)\} \| \leq M, \ \forall \ j, \ k \tag{3.22}$$

Therefore, from (3.21)-(3.22), there exists $\delta_2 > 0$ such that

$$P_x(k|k;j|j) \leq \delta_2 I, \quad \forall \ j, \ k \tag{3.23}$$

Hence, from Lemma 3.1, we have

$$0 < \delta_1 I \leq P_x(k|k;j|j) \leq \delta_2 I, \ \forall \ k \geq N > 0 \text{ and } \forall \ j \qquad \text{Q.E.D.}$$

Lemma 3.3: Given the dynamical system (1.1)-(1.2), the observations z_k and z_n become uncorrelated as $|k-n| \to \infty$ for all $k,n \geq 0$.

Proof: It is sufficient to show that states x_k and x_n are asymptotically uncorrelated. From (1.1), it follows that

$$E\{x_k x_n^T|I_j(k)\} = A^k V_x(0)(A^T)^n + \sum_{i=0}^{n-1} A^{k-i-1} BQ_j(i)B^T(A^T)^{n-i-1} \ ^{\dagger}$$

where $V_x(0) \triangleq E\{x_0 x_0^T|I_j(k)\}$, $k \geq n$, $j = 1,2,\ldots,s^{k+1}$.

$$\therefore E\{x_k x_n^T|I_j(k)\} = A^{k-n}(A^n V_x(0)(A^T)^n) + \sum_{i=0}^{n-1} (A^{n-i-1} BQ_j(i)B^T(A^T)^{n-i-1})$$

$$\leq A^{k-n}(A^n V_x(0)(A^T)^n) + \sum_{i=0}^{n-1} A^{k-n}(A^{n-i-1} \cdot B \cdot QU \cdot B^T(A^T)^{n-i-1}), \forall j$$

$^{\dagger}\{Q_j(i)\}_{i=0}^{k}$ is the covariance sequence corresponding to $\{w_i\}_{i=0}^{k}$ in the state sequence $I_j(k)$.

It follows that

$E\{x_k x_n^T | I_j(k)\} \to 0$ as $|k-n| \to \infty$ uniformly in $I_j(k)$ since, by condition (H1), A has all its eigenvalues inside the unit circle. Hence $E\{x_k x_n^T\} \to 0$ as $|k-n| \to \infty$. Q.E.D.

Remark 3.2: Since, given $I_j(k)$, $\{z_k\}$ is a Gaussian time series, it follows from the proof of Lemma 3.3 that z_k, z_n become asymptotically, conditionally independent, uniformly in j, given the state sequence $I_j(k)$, $j = 1,2,\ldots,S^{k+1}$; $k > n$.

Lemma 3.4: Let the dynamical system (1.1)-(1.2) begin at initial time $k_0 = -\infty$ (instead of $k_0 = 0$). Then, for $k \geq k_1 > -\infty$, the observation sequence $\{z_k\}$ is stationary.

Proof: Let $Z_{n,k_1} = \{z_k\}_{k=k_1}^{n}$, and let its joint probability density function, assuming π to be the transition matrix, be given by

$$f_{Z_{n,k_1}}(\underline{\alpha}|\pi) = \sum_j f_{Z_{n,k_1}|I_j(n,k_1)}(\underline{\alpha}) P(I_j(n,k_1)|\pi) \tag{3.24}$$

where $\underline{\alpha} = \{\alpha_i\}_{i=1}^{n-k_1+1}$ denotes value of Z_{n,k_1}; $I_j(n,k_1)$ represents a specific state sequence starting at time k_1 and ending at n, $j = 1,2,\ldots,S^{n-k_1+1}$; and $f_{Z_{n,k_1}|I_j(n,k_1)}(\cdot)$ is the density function of Z_{n,k_1} given $I_j(n,k_1)$. We need to show that

$$f_{Z_{n,k_1}}(\underline{\alpha}|\pi) = f_{Z_{n+\ell,k_1+\ell}}(\underline{\alpha}|\pi), \ \forall n \geq k_1; \ell \geq 0; \ \forall \underline{\alpha}. \tag{3.25}$$

The right side of (3.25) may be expressed as

$$f_{Z_{n+\ell,k_1+\ell}}(\underline{\alpha}|\pi) = \sum_j f_{Z_{n+\ell,k_1+\ell}|I_j(n+\ell,k_1+\ell)}(\underline{\alpha}) P(I_j(n+\ell,k_1+\ell)|\pi) \tag{3.26}$$

where $I_j(n+\ell,k_1+\ell)$ is a shifted version of $I_j(n,k_1)$. From condition (H4) in Section 3.4, it follows that

$$P(I_j(n+\ell,k_1+\ell)|\pi) = P(I_j(n,k_1)|\pi),\ j = 1,2,\ldots,S^{n-k_1+1},\ \ell \geq 0. \quad (3.27)$$

Furthermore,

$$f_{Z_{n,k_1}|I_j(n,k_1)}(\underline{\alpha}) = f_{z_{k_1}|I_j(k_1,k_1)}(\alpha_1) \times \prod_{m=k_1+1}^{n} f_{z_m|Z_{m-1,k_1},I_j(m,k_1)}(\alpha_{m+1-k})|\alpha_{m-k_1},\ldots,\alpha_1) \quad (3.28)$$

where $I_j(m,k_1)$ is a subsequence of $I_j(n,k_1)$ and α_1 is the value of z_{k_1+i-1}. From Lemma 3.3 and condition (H4) (and Remark 3.1) in Section 3.4, it follows that

$$f_{z_{k_1}|I_j(k_1,k_1)}(\alpha_1) \equiv f_{z_{k_1+\ell}|I_j(k_1+\ell,k_1+\ell)}(\alpha_1) \quad \forall\ \ell \geq 0 \quad (3.29)$$

with similar results for other terms in (3.28). Consequently, it follows that

$$f_{Z_{n,k_1}|I_j(n,k_1)}(\underline{\alpha}) \equiv f_{Z_{n+\ell,k_1+\ell}|I_j(n+\ell,k_1+\ell)}(\underline{\alpha}) \quad (3.30)$$

which proves that (3.25) holds. Q.E.D.

Lemma 3.5: Let condition (H3) in Section 3.4 be satisfied. Then the Kalman filters matched to state sequences $I_j(k)$ (that is, Kalman filters designed assuming state sequences $I_j(k)$ to be true in model (1.1)-(1.2)), $j = 1,2,\ldots,S^{k+1}$, are uniformly asymptotically stable, uniformly in k and in j.

Proof: The Kalman filter matched to the state sequenc $I_j(k)$ is given by the following equations

$$\hat{x}_j(k|k) = \Psi_j(k,k-1)\hat{x}_j(k-1|k-1) + K_j(k)z_k \quad (3.31)$$

$$K_j(k) = P_x(k|k-1;j|j)C^T[CP_x(k|k-1;j|j)C^T + R_j(k)]^{-1} \quad (3.32)$$

$$P_x(k|k;j|j) = \Psi_j(k,k-1)P_x(k-1|k-1;j|j)\Psi_j^T(k,k-1) + F_j(k) \quad (3.33)$$

$$F_j(k) = [I-K_j(k)C]BQ_j(k)B^T[I-K_j(k)C]^T + K_j(k)R_j(k)K_j^T(k) \qquad (3.34)$$

where $j = 1,2,\ldots,S^{k+1}$; $\Psi_j(k,k-1) = [I-K_j(k)C]A$ and

$\Psi_j(k,\ell) = \Psi_j(k,k-1)\Psi_j(k-1,k-2)\ldots\Psi_j(\ell+1,\ell)$.

The Kalman filter given by equations (3.31)-(3.34) is uniformly asymptotically stable in the large if there exist real scalar functions $V_j(\hat{x}_j(k|k),k)$, $\gamma_{1j}(\|\hat{x}_j(k|k)\|)$, $\gamma_{2j}(\|\hat{x}_j(k|k)\|)$, and $\gamma_{3j}(\|\hat{x}_j(k|k)\|)$ such that for some finite $N \geq 0$

$$0 < \gamma_{1j}(\|\hat{x}_j(k|k)\|) \leq V_j(\hat{x}_j(k|k),k) \leq \gamma_{2j}(\|\hat{x}_j(k|k)\|), \quad \hat{x}_j(k|k) \neq 0 \qquad (3.35)$$

$$V_j(\hat{x}_j(k|k),k) - V_j(\hat{x}_j(k-N|k-N),k-N) \leq \gamma_{3j}(\|\hat{x}_j(k|k)\|) < 0$$

$$k \geq N, \ \hat{x}_j(k|k) \neq 0 \qquad (3.36)$$

and, in addition,

$$\gamma_{1j}(0) = \gamma_{2j}(0) = 0, \qquad \lim_{\rho \to \infty} \gamma_{1j}(\rho) = \infty, \qquad (3.37)$$

(see [28], [18], [30]). It has been shown in [18], [28] that an appropriate Lyapunov function is given by

$$V_j(\hat{x}_j(k|k),k) = \hat{x}_j^T(k|k)P_x^{-1}(k|k;j|j)\hat{x}_j(k|k) \qquad (3.38)$$

From Lemma 3.1 and Lemma 3.2 we can find

$$\gamma_1(\|\hat{x}_j(k|k)\|) \leq \gamma_{1j}(\|\hat{x}_j(k|k)\|) \quad \forall\ j$$

$$\gamma_2(\|\hat{x}_j(k|k)\|) \geq \gamma_{2j}(\|\hat{x}_j(k|k)\|) \quad \forall\ j$$

to satisfy (3.35), (3.37) for all j, $j = 1,2,\ldots,S^{k+1}$. Further, following the solution given in [18] and using Lemma 3.1 and Lemma 3.2, we can find $\gamma_3(\cdot)$ such that

$$\gamma_{3j}(\|\hat{x}_j(k|k)\|) \leq \gamma_3(\|\hat{x}_j(k|k)\|) < 0 .$$

Then the Kalman filters matched to $I_j(k)$ are uniformly asymptotically stable, uniformly in k and in j (see Theorems 4.2 and 4.3 in [68]). That is, given $\varepsilon > 0 \ni \delta(\varepsilon) > 0$ and $T(\varepsilon,\delta) > 0 \ni \|\hat{x}_j(k|k)\| \leq \varepsilon$ for all $k \geq \ell + T(\varepsilon,\delta)$ if $\|\hat{x}_j(\ell|\ell\| \leq \delta$, for all $\ell \geq 0$ and $\forall$ j, where we consider the system given by

$$\hat{x}_j(k|k) = \Psi_j(k|k-1)\hat{x}_j(k-1|k-1) \tag{3.39}$$

Q.E.D.

Remark 3.3: An important consequence of Lemma 3.5 is that (see Theorem 3 in [30]) there exist positive constants c_1, c_2 such that

$$\|\Psi_j(k,\ell)\| \leq c_1 \exp[-c_2(k-\ell)] \quad \forall \ j;\ k \geq \ell \tag{3.40}$$

Further, the effect of initial statistic P_0 and estimate $\hat{x}(0|0)$ is forgotten as more and more data are processed. Moreover, this happens uniformly in all possible state sequences.

Lemma 3.6: (i) Let the dynamical system (1.1)-(1.2) begin at initial time $k_0 = \ell$. Then

$$\lim_{\ell \to -\infty} f_{z_k|Z_{k-1,\ell}}(\alpha_k|\underline{\alpha}_{k-1,\ell},\pi) = f_{z_k|Z_{k-1,-\infty}}(\alpha_k|\underline{\alpha}_{k-1,-\infty},\pi), \text{ a.e. } P^Z_{\pi_t}$$

where $f_{z_k|Z_{k-1,\ell}}(\cdot|\cdot,\pi)$ is the density function of z_k given

$Z_{k-1,\ell} = \{z_n\}_{n=\ell}^{k-1}$ assuming the transition matrix π to be true with $\underline{\alpha}_{k-1,\ell}$ appropriately defined.

(ii) Let the dynamical system (1.1)-(1.2) begin at initial time $k_0 = -\infty$. Then the random sequence $\{f_{z_k|Z_{k-1}}(z_k|Z_{k-1},\pi)\}_{k=0}^{\infty}$ becomes asymptotically stationary, where $Z_{k-1} = \{z_n\}_{n=0}^{k-1}$.

Proof: (i) We have

$$f_{z_k|Z_{k-1,\ell}}(\alpha_k|\underline{\alpha}_{k-1,\ell},\pi) = \sum_{I_j(k,\ell)} f_{z_k|Z_{k-1,\ell},I_j(k,\ell)}(\alpha_k|\underline{\alpha}_{k-1,\ell},I_j(k,\ell)) \times P(I_j(k,\ell)|\underline{\alpha}_{k-1,\ell},\pi) \quad (3.41)$$

where $I_j(k,\ell)$ is the state sequence beginning at ℓ and ending at k, $j = 1,2,\ldots,S^{k-\ell+1}$; $k \geq \ell$. Further, $f_{z_k|Z_{k-1,\ell},I_j(k,\ell)}(\alpha_k|\underline{\alpha}_{k-1,\ell},I_j(k,\ell))$ is a Gaussian density function with mean $E\{z_k|\underline{\alpha}_{k-1,\ell},I_j(k,\ell)\}$, and covariance given by an equation similar to that for $P_z(k|k;j|j)$. From Lemma 3.5, we have

$$\lim_{\ell \to -\infty} f_{z_k|Z_{k-1,\ell},I_j(k,\ell)}(\alpha_k|\underline{\alpha}_{k-1,\ell},I_j(k,\ell)) = f_{z_k|Z_{k-1,-\infty},I_j(k,-\infty)}(\alpha_k|\underline{\alpha}_{k-1,-\infty},I_j(k,-\infty)) \quad (3.42)$$

uniformly in j, for all bounded infinite sequences $\underline{\alpha}_{k,-\infty}$. Now define

$$\Delta_{\ell,m} \triangleq f_{z_k|Z_{k-1,\ell}}(\alpha_k|\underline{\alpha}_{k-1,\ell},\pi) - f_{z_k|Z_{k-1,m}}(\alpha_k|\underline{\alpha}_{k-1,m},\pi)$$

From (3.42) given $\epsilon_1 > 0$ $\exists$ $N_1(\epsilon_1) = N_1$ such that

$$|f_{z_k|Z_{k-1,\ell},I_j(k,\ell)}(\alpha_k|\underline{\alpha}_{k-1,\ell},I_j(k,\ell)) - f_{z_k|Z_{k-1,k-N_1(\epsilon_1)},I_j(k,k-N_1(\epsilon_1))}(\alpha_k|\underline{\alpha}_{k-1,k-N_1(\epsilon_1)},I_j(k,k-N_1(\epsilon_1)))| < \epsilon_1$$

for $\ell < k-N(\epsilon_1)$, uniformly in j, $j = 1,2,\ldots,S^{k-\ell+1}$. Therefore

$$|\Delta_{\ell,m} - \sum_{I_j(k,k-N_1)} [f_{z_k|Z_{k-1,k-N_1},I_j(k,k-N_1)}(\alpha_k|\underline{\alpha}_{k-1,k-N_1},I_j(k,k-N_1)) \times \{P(I_j(k,k-N_1)|\underline{\alpha}_{k-1,\ell},\pi) - P(I_j(k,k-N_1)|\underline{\alpha}_{k-1,m},\pi)\}]| \leq 2\epsilon_1, \quad \forall\ \ell,m \leq k-N_1(\epsilon_1) \quad (3.43)$$

From condition (H5), given $\epsilon_2 > 0$ $\exists$ $N_2(\epsilon_1,\epsilon_2) = N_2$ such that

$$|P(I_j(k,k-N_1(\epsilon_1))|\underline{\alpha}_{k-1,\ell},\pi) - P(I_j(k,k-N_1(\epsilon_1))|\underline{\alpha}_{k-1,m},\pi)| \leq \epsilon_2, \text{a.e. } P^Z_{\pi_t} \tag{3.44}$$

for $\ell,m < k-N_2(\epsilon_1,\epsilon_2)$, and for all j. (Note that there are finite $(S^{N_1(\epsilon_1)+1})$ number of state sequences of length $N(\epsilon_1)+1$.) Therefore, from (3.43), (3.44) and Lemma 3.2, we have

$$|\Delta_{\ell m}| \leq 2\epsilon_1 + S^{N_1(\epsilon_1)+1} \times \epsilon_2 \times M \text{ for } \ell,m < k-N,\ N = \max(N_1,N_2),$$

where

$$|f_{z_k|Z_{k-1,k-N_1},I_j(k,k-N_1)}(\alpha_k|\underline{\alpha}_{k-1,k-N_1},I_j(k,k-N_1))| \leq M \quad \forall\, k, j, N \text{ (Lemma 3.2)}.$$

Noting that the space of all bounded, infinite sequences carry probability one, it follows that Lemma 3.6(i) holds. Q.E.D.

(ii) From Lemma 3.4, the observation sequence $\{z_k\}_{k=0}^{\infty}$ is stationary. Extend this single-ended process into a double-ended stationary process $\{z'_k\}_{k=-\infty}^{\infty}$ such that $\{z_k\}_{k=0}^{\infty}$ and $\{z'_k\}_{k=0}^{\infty}$ have the same distribution (see Proposition 6.5 in [10]). To avoid cumbersome notation, denote $\{z'_k\}_{k=-\infty}^{\infty}$ by $\{z_k\}_{k=-\infty}^{\infty}$. Define $Z_{n,\ell} = \{z_k\}_{k=\ell}^{n}$, $n \geq \ell$. Now for any finite ℓ, $f_{z_k|Z_{k-1,\ell}}(\cdot,\pi)$ is a measurable function. Since $f_{z_k|Z_{k-1,\ell}}(\cdot,\pi)$ converges pointwise a.e. to $f_{z_k|Z_{k-1,-\infty}}(\cdot,\pi)$ as $\ell \to -\infty$, it follows that $f_{z_k|Z_{k-1,-\infty}}(\cdot,\pi)$ is also measurable mapping from $(\prod_{s=1}^{m}(R^{\infty},\mathcal{B}^{\infty})) \to (R,\mathcal{B})$[†] (see [5, Theorem 1.5.4]). Then from Proposition 6.6 in Breiman [10], $f_{z_k|Z_{k-1,-\infty}}(z_k|Z_{k-1,-\infty},\pi)$ is stationary, k = 1,2,... Q.E.D.

[†] R is the real line and $\mathcal{B}$ is the corresponding Borel σ-field. R^{∞} is the space of infinite sequences of real numbers and $\mathcal{B}^{\infty}$ is the corresponding Borel σ-field.

Remark 3.4: (i) From Lemma 3.1 and Lemma 3.2 we have

$$0 \leq f_{z_k|Z_{k-1}}(z_k|Z_{k-1},\pi) \leq M_1 < \infty, \forall\ k \geq 0.$$

Therefore, from the dominated convergence theorem (Ash [5]),

$\lim_{k \to \infty} f_{z_k|Z_{k-1}}(z_k|Z_{k-1},\pi)$ is integrable.

(ii) Since $\ell n(\cdot)$ is a continuous function, it readily follows that the random sequence $\{\ell n\ f_{z_k|Z_{k-1}}(z_k|Z_{k-1},\pi)\}_{k=0}^{\infty}$ becomes asymptotically stationary.

Lemma 3.7: Consider the double-ended stationary process $\{z_k\}_{k=-\infty}^{\infty}$ as defined in the proof of Lemma 3.6. Define $Z_{n,\ell} = \{z_k\}_{k=\ell}^{\infty}$, $n \geq \ell$. Then

$\ell n\ f_{z_k|Z_{k-1,-\infty}}(z_k|Z_{k-1,-\infty},\pi)$ is integrable, $k = 0,1,\ldots$.

Proof: In Lemma 3.6 we showed that

$$\lim_{\ell \to -\infty} f_{z_k|Z_{k-1,\ell}}(z_k|Z_{k-1,\ell},\pi) = f_{z_k|Z_{k-1,-\infty}}(z_k|Z_{k-1,-\infty},\pi) \text{ a.e.}$$

Now

$$f(z_k|Z_{k-1,\ell},\pi) = \sum_{I_j(k,\ell)} f(z_k|Z_{k-1,\ell},I_j(k,\ell))P(I_j(k,\ell)|Z_{k-1,\ell},\pi) \tag{3.45}$$

where we have dropped subscripts on $f(\cdot)$. Furthermore, since $f(\cdot)$ is Gaussian:

$$f(z_k|Z_{k-1,\ell},I_j(k,\ell)) = [(2\pi)^{m/2}|P_{z\ell}(k|k-1;j|j)|^{\frac{1}{2}}]^{-1}$$

$$\times \exp[-\tfrac{1}{2}(z_k-\hat{z}_\ell(k|k-1;j))^T P_{z\ell}^{-1}(k|k-1;j|j)(z_k-\hat{z}_\ell(k|k-1;j))] \tag{3.46}$$

$$j = 1,2,\ldots,S^{k-\ell+1}$$

where subscript ℓ on $\hat{z}_\ell(k|k\text{-}1;j)$ and $P_{z\ell}(k|k\text{-}1;j|j)$ indicates that these quantities have been computed for the observation $Z_{n\ell}$. Then

$$\begin{aligned} \ell n\ f(z_k|Z_{k-1,\ell},I_j(k,\ell)) &= -\frac{m}{2}\ell n2\pi - \frac{1}{2}\ell n|P_{z\ell}(k|k\text{-}1;j|j)| \\ &\quad - \frac{1}{2}(z_k-\hat{z}_\ell(k|k\text{-}1;j))^T P_{z\ell}^{-1}(k|k\text{-}1;j|j)(z_k-\hat{z}_\ell(k|k\text{-}1;j)) \\ &\geq -\frac{m}{2}\ell n2\pi - \frac{1}{2}\ell n\beta_2 - \frac{1}{2}\beta_1^{-1}\cdot\|z_k-\hat{z}_\ell(k|k\text{-}1;j)\|^2 \end{aligned} \tag{3.47}$$

where, from Lemma 3.2, there exists β_1, $\beta_2 > 0$ such that

$$\beta_1 I \leq P_{z\ell}(k|k\text{-}1;j|j) \leq \beta_2 I,\ \ell \leq k\text{-}N;\ \forall\ k,j \tag{3.48}$$

In a manner similar to (3.31) we have

$$\hat{x}_{j\ell}(k|k) = \Psi_j(k,\ell)\hat{x}_{j\ell}(\ell|\ell) + \sum_{i=\ell}^{k-1}\Psi_j(k,i+1)K_{j\ell}(i)z_i \tag{3.49}$$

where

$$\|\Psi_j(k,\ell)\| \leq c_1\exp[-c_2(k-\ell)],\ c_1,c_2 > 0 \text{ (see Lemma 3.5)};\ \forall\ j,$$

which with (3.49) results in

$$\begin{aligned} \hat{z}_\ell(k|k\text{-}1;j) &= C\,\hat{x}_{j\ell}(k|k\text{-}1) = CA\hat{x}_{j\ell}(k\text{-}1|k\text{-}1) \\ &= CA\Psi_j(k,\ell)\hat{x}_{j\ell}(\ell|\ell) + \sum_{i=\ell}^{k-1} CA\Psi_j(k,i+1)K_{j\ell}(i)z_i\ . \end{aligned} \tag{3.50}$$

Therefore, the norm of the prediction error satisfies

$$\begin{aligned}
\|z_k-\hat{z}_\ell(k|k-1;j)\|^2 &\le \|z_k\|^2+2\cdot\|z_k\|\cdot\|\hat{z}_\ell(k|k-1;j)\|+\|\hat{z}_\ell(k|k-1;j)\|^2 \\
&\le \|z_k\|^2+2\cdot\|z_k\|\cdot\|CA\|\cdot c_1[\exp(-c_2(k-\ell))\cdot\|\hat{x}_{j\ell}(\ell|\ell)\| \\
&+ \sum_{i=\ell}^{k-1}\exp(-c_2(k-i))\cdot\|K_{j\ell}(i)\|\cdot\|z_i\|] \\
&+ \|CA\|^2\cdot c_1^2\exp[-2c_2(k-\ell)]\cdot\|\hat{x}_{j\ell}(\ell|\ell\|^2 \\
&+ \|CA\|^2\cdot c_1^2\cdot\sum_{i=\ell}^{k-1}\|\hat{x}_{j\ell}(\ell|\ell)\|\cdot\|K_{j\ell}(i)\|\cdot\|z_i\| \\
&\cdot \exp[-c_2(k-\ell)]\exp[-c_2(k-i-1)] \\
&+ \sum_{i=\ell}^{k-1}\sum_{m=\ell}^{k-1}\|CA\|^2\cdot\|K_{j\ell}(i)\|\cdot\|K_{j\ell}(m)\|\cdot\|z_i\|\cdot\|z_m\|\cdot c_1^2 \\
&\cdot \exp(-c_2(k-i-1))\cdot\exp(-c_2(k-m-1)) \qquad (3.51)
\end{aligned}$$

Noting that $\|K_{j\ell}(i)\| < M_3 < \infty$, $\forall\ \ell$, j, i and $\|CA\| < M_4 < \infty$, we have

$$\begin{aligned}
\|z_k-\hat{z}_\ell(k|k-1;j)\|^2 &\le \|z_k\|^2+M_6\sum_{i=-\infty}^{k-1}\|z_k\|\cdot\|z_i\|\cdot\exp(-c_2(k-i)) \\
&+ M_7\sum_{i=-\infty}^{k-1}\sum_{m=-\infty}^{k-1}\|z_i\|\cdot\|z_m\|\cdot\exp(-c_2(k-i-1))\exp(-c_2(k-m-1)) \\
&\triangleq g(Z_{k,-\infty}),\ \forall\ \ell,j \qquad (3.52)
\end{aligned}$$

where M_6 and M_7 are appropriate positive constants.

Now we claim that $|E_t\{g(Z_{k,-\infty})\}| < \infty$ where the subscript t indicates that the expectation is taken w.r.t. distribution corresponding to the true stochastic matrix π_t. In order to prove this claim, note that

$$E_t\{\|z_k\| \cdot \|z_i\|\} \le [E_t\{\|z_k\|^2\}] = M_8 < \infty, \ \forall\ k,i.$$

Hence from (3.52) it follows that

$$0 \le E_t\{g(Z_{k,-\infty})\} \le M_8 + M_6 \cdot M_8 \cdot \sum_{i=-\infty}^{k-1} \exp(-c_2(k-1))$$

$$+M_7 \cdot M_8 \left(\sum_{i=-\infty}^{k-1} \exp(-c_2(k-i-1)) \right)^2 < \infty. \tag{3.53}$$

Therefore, from (3.47), (3.52) and (3.53), one obtains

$$\ell n\ f(z_k | Z_{k-1,\ell}, I_j(k,\ell)) \ge h(Z_{k,-\infty})\ \forall\ \ell, j \tag{3.54}$$

where

$$h(Z_{k,-\infty}) = -\frac{m}{2}\ \ell n\ 2\pi - \frac{1}{2}\ \ell n\ \beta_2 - \frac{1}{2}\beta_1^{-1}\ g(Z_{k,-\infty}) \tag{3.55}$$

and $$|E_t\{h(Z_{k,-\infty})\}| < \infty\ . \tag{3.56}$$

The use of (3.54) and (3.55) yields

$$\ell n\ f(z_k | Z_{k-1,\ell}, \pi) \ge h(Z_{k,-\infty}) \tag{3.57}$$

Further, from (3.45), (3.46) and (3.48), we have

$$\ell n\ f(z_k | Z_{k-1,\ell}, \pi) \le -\frac{m}{2}\ \ell n\ 2\pi - \frac{1}{2}\ \ell n\ \beta_1 = \text{constant}, \ \forall\ \ell \tag{3.58}$$

Then from Lemma 3.6(i), Equations (3.56)-(3.58) and Fatou's Lemma (Ash [5]), the desired result follows. Q.E.D.

Remark 3.5: Consider the single-ended stationary process $\{z_i\}_{i=0}^{+\infty}$ resulting from the dynamical system (1.1)-(1.2). Let $Z_k = \{z_i\}_{i=0}^{k}$. Then from the proof of Lemma 3.7, we have

$$|E_t\{\ell n\ f(z_k|Z_{k-1},\pi)\}| \leq M_{10} < \infty, k = 0,1,2,... \tag{3.59}$$

and

$$E_t\{|\ell n\ f(z_k|Z_{k-1},\pi)|\} \leq M_{11} < \infty, k = 0,1,2,... \quad . \tag{3.60}$$

If we now define

$$y_k = \ell n\ f(z_k|Z_{k-1},\pi) - E_t\{\ell n\ f(z_k|Z_{k-1},\pi)\} \tag{3.61}$$

then $\{y_k\}_{k=0}^{\infty}$ is an asymptotically stationary (Lemma 3.6), zero-mean random sequence.

Lemma 3.8: $P\{n^{-1}|y_1+y_2+...+y_n| > \epsilon\} \to 0$, for any $\epsilon > 0$, as $n \to \infty$.

Proof: From Feller Vol. II[20, p. 240], it is sufficient to show that

$$E_t\{y_\ell y_k\} \to 0 \text{ as } |k-\ell| \to \infty \ .$$

Let for example $k > \ell$, then

$$E_t\{y_\ell y_k\} = \sum_{I_j(k)} E_t\{y_\ell y_k|I_j(k)\} \times P(I_j(k)|\pi_t),\ j = 1,2,...,S^{k+1} \ .$$

From Lemma 3.3, Remark 3.2, Lemma 3.6(i) and Lemma 3.7, it follows that

$$\lim_{|k-\ell|\to\infty} E_t\{y_\ell y_k|I_j(k)\} \to 0 \text{ uniformly in } j.$$

Therefore,

$$E_t\{y_\ell y_k\} \to 0 \text{ as } |k-\ell| \to \infty. \qquad \text{Q.E.D.}$$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: According to Tse and Anton [63], the following 6 conditions† are sufficient for the weak consistency of the ML estimate to hold:

†Condition (AC6) is given in Tse [64]. It was overlooked in [63].

(AC1) $f(Z_n|\pi)$ is measurable in Z_n w.r.t. measure $\mu_n(dZ_n) \triangleq f(Z_n|\pi_t)dZ_n$ where π_t is the true transition matrix, and $Z_n = \{z_i\}_{i=0}^n$.

(AC2) For each $\pi \in \Omega'$ and all $n = 1,2,\ldots$, $E_t\{\ell n\, f(z_n|Z_{n-1},\pi)\} < \infty$, where $\Omega' = \{\pi_1,\ldots,\pi_M\}$. Also, $E_t\{|\ell n\, f(z_n|Z_{n-1},\pi_t)|\} < \infty$, $n=1,2,\ldots$.

(AC3) For each $\pi \in \Omega'$,

$$\text{var}\{n^{-1} \sum_{k=1}^{n} \ell n\, f(z_k|Z_{k-1},\pi)\} \to 0 \text{ as } n \to \infty.$$

This can be replaced by assumption (AC3').

(AC3') The observations $\{z_i\}$ are asymptotically uncorrelated.

(AC4) Define the set $\mathcal{B}_n(\pi) \triangleq \{Z_n: f(Z_n|\pi) = 0\}$. Then for all $\pi,\pi' \in \Omega'$, we have

$$\mathcal{B}_n(\pi) = \mathcal{B}_n(\pi') \;\forall\; n = 1,2,\ldots .$$

(AC5) For all $\pi_q \in \Omega'$, $\pi_q \neq \pi_t$, there exists an infinite set $G \subset I^+$, such that the inequality

$$f(z_n|Z_{n-1},\pi_q) \neq f(z_n|Z_{n-1},\pi_t)$$

holds with nonzero probability w.r.t. π_t uniformly in $n \in G$. (Note that I^+ = set of nonnegative integers.)

(AC6) Define $u_n = \ell n\, f(z_n|Z_{n-1},\pi_q) - \ell n\, f(z_n|Z_{n-1},\pi_t)$, $\pi_q \neq \pi_t$, and $\pi_q,\pi_t \in \Omega'$. Then $E_t\{u_n\} < 0$ uniformly in n.

Now, conditions (AC1) and (AC4) are obviously satisfied since we are dealing with Gaussian mixtures. Lemma 3.7 and Remark 3.5 satisfy condition (AC2). Lemma 3.8 replaces condition (AC3) since (AC3) is used in Tse and Anton [63] as a sufficient condition for the weak law of large

numbers to hold for the sequence $\{\ell n\ f(z_k|Z_{k-1},\pi)\}$.† Condition (H6) and Lemma 3.6 satisfy (AC5) (see also Remark 3.6). It remains to show that condition (AC6) holds. From Lemma 3.6, $u_n \to u_\infty$ pointwise a.e., where

$$u_\infty = \lim_{n \to \infty} [\ell n\ f(z_n|Z_{n-1},\pi_t) - \ell n\ f(z_n|Z_{n-1},\pi_t)] \tag{3.62}$$

and

$$E_t\{u_\infty\} < \infty\ .$$

From Equation (46) in [63], we have

$$E_t\{u_n\} < \ell n E_t\{\exp(u_n)\},\ n = 1,2,\ldots,\infty. \tag{3.63}$$

Also, from Equation (47) in [63], one obtains

$$E_t\{\exp(u_n)\} = 1,\ n = 1,2,\ldots\ . \tag{3.64}$$

Using Fatou's lemma [5], we obtain

$$\lim_{n \to \infty}\ \ell n\ E_t\{\exp(u_n)\} \geq \ell n\ E_t\{\exp(u_\infty)\} \tag{3.65}$$

Following the procedure given in the proof of Lemma 3.7, and using Fatou's lemma (or the dominated convergence theorem) [5], we have

$$\lim_{n \to \infty}\ E_t\{u_n\} = E_t\{u_\infty\}\ . \tag{3.65}$$

Hence, from (3.63)-(3.65), we have

$$E_t\{u_\infty\} < 0\ . \tag{3.67}$$

Therefore, condition (AC6) holds.

Thus we have proved Theorem 3.1. Q.E.D.

Using Theorem 3.1, we can easily establish the following:

---

†Note that Lemma 3.3 satisfies condition (AC3'). However, it is not clear how (AC3') can replace (AC3).

Theorem 3.2: Suppose that the hypotheses of Theorem 3.1 hold. Suppose that every π in the finite set $\{\pi_1,\pi_2,\text{--},\pi_M\}$, which includes the correct value π_t, is assigned nonzero a priori probability. Then

$$\lim_{n \to \infty} P(\pi_q|Z_n) = 1 \quad \text{for} \quad \pi_q = \pi_t$$

$$= 0 \quad \text{for} \quad \pi_q \neq \pi_t$$

in probability

Proof: It follows immediately from Theorem 1 in Yamada [71].

Remark 3.6: Condition (H6) is needed to satisfy an identifiability condition [63] (see condition (AC5)). It is a sufficient condition for two transition matrices π_q and π_t to be "resolvable" [63] through maximum likelihood identification. From Lemma 3.6 $f(z_n|Z_{n-1},\pi_q)$ becomes stationary for large n and all π_q's. So either condition (AC5) holds or $\lim_{n \to \infty} f(z_n|Z_{n-1},\pi_q) = \lim_{n \to \infty} f(z_n|Z_{n-1},\pi_t)$ a.e. . It appears that if π_q and π_t are such that $\pi_q \neq \pi_t$ then (H6) is satisfied, because then $\lim_n \pi_q^n \neq \lim_n \pi_t^n$, consequently, the observation processes $\{z_k\}$ corresponding to these π's become, in the limit, two stationary processes with different statistics (see Lemma 3.4). However, a proof of this is not yet available. Under certain situations lack of resolvability is not a problem since identification per se is not the objective. For example, consider the problem where switchings occur only in the driving noise statistics of model (1.1)-(1.2). Then ML estimate converges to a π_q such that $\hat{x}(k|k;\pi_q) = \hat{x}(k|k;\pi_t)$ for large k. This is acceptable if state estimation is the sole objective.

Remark 3.7: Condition (H5) is needed in Lemma 3.6 to prove the pointwise convergence of $f(z_n|Z_{n-1,\ell},\pi_q)$ a.e., as $\ell \to -\infty$, for all admissible π_q's,

where system model (1.1)-(1.2) is assumed to have started at initial time $k_o = \ell$ and $Z_{n,\ell} = \{z_k, \ell \leq k \leq n\}$. So far we have been unable to find a proof for this. However, from heuristic considerations, this seems to be true. Note that system model (1.1) is assumed to be stable. Lemma 3.2 ensures that $f(z_n|Z_{n-1,\ell},\pi_q)$ is well defined for all $Z_{n,\ell}$, $\ell \geq -\infty$. Since the system is stable and the Markov chain governing switchings in the noise statistics is ergodic, it follows that the initial conditions (initial system state value x_ℓ, initial noise values w_ℓ and v_ℓ, and initial Markov chain state i_ℓ) have no effect on the asymptotic behavior of the system. So $f(z_n|Z_{n-1,\ell}, \pi_q)$ should converge pointwise to $f(z_n|Z_{n-1,-\infty}, \pi_q)$ as $\ell \to -\infty$, for all bounded, infinite sequences $\{---,z_{-1},z_0,---,z_n\}$, $n > -\infty$. For the same reasons, (H5) must hold. From the Martingale theory [5], it follows that

$$\lim_{k_0 \to \infty} P(I_j(n,k)|\pi_q,Z_{n,k_0}) = P(I_j(n,k)|\pi_q,Z_{n,-\infty}) \quad \text{a.e. } P^Z_{\pi_q} .$$

It then follows that there exists a set of bounded, infinite sequences $\{---,z_{-1},z_0,--,z_n\}$ for which

$$\lim_{k_0 \to -\infty} P(I_j(n,k)|\pi_q,Z_{n,k_0}) = P(I_j(n,k)|\pi_q,Z_{n,-\infty}) .$$

However, we have been unable to show (due to the complicated expression for $P(I_j(n,k)|\pi_q,Z_{n,k_0})$) that this holds for every bounded, infinite observation sequence. Nevertheless, it does serve to indicate that under intuitively reasonable condition convergence in Theorem 3.1 is attained.

Remark 3.8: Theorems 3.1 and 3.2 prove the convergence of the optimal algorithm given in Section 3.2. But in practice we use approximations given in Section 3.3. From properties P1) (see Eqn. (B8)) and P3) in

Appendix B, it follows that $f(Z_k|\pi_q)$ can be approximated arbitrarily closely by taking sufficiently large N, using the random sampling approach. So, in a sense, convergence of the random sampling algorithm follows.

Remark 3.9: Results can be extended in a straightforward fashion to the case of non-zero mean noise; the details are omitted.

3.5. Convergence in Performance

Now we present some results on the convergence of the performance of the optimal estimator to that of an estimator operating with complete knowledge of the true transition matrix. We make use of some results of Hilborn and Lainiotis [24] after some modifications. First we need a preliminary result.

Lemma 3.9: Suppose that conditions (H1)-(H4) are satisfied. Given a positive definite symmetric matrix F, there exists a bound $M_1 < \infty$ such that

$$E\{[\hat{x}(k|k;\pi_q)F\ \hat{x}^T(k|k;\pi_r)]^2\} \leq M_1$$

for all k, where $\hat{x}(k|k;\pi_q) = E\{x_k|Z_k,\pi_q\}$ and $\pi_q,\pi_r \in \Omega$.

Proof:

Equation (3.6) implies that

$$\|\hat{x}(k|k;\pi_q)\| \leq \sum_j \|\hat{x}_j(k\ k)\| \times P(I_j(k)|Z_k,\pi_q) \tag{3.68}$$

which with Equation (3.31), and after noting that $\|K_j(i)\| \leq M_3 < \infty$, $\forall j,i$, yields

$$\|\hat{x}_j(k|k)\| \leq \sum_{i=-\infty}^{k-1} M_3 \times \|z_i\| \times c_1 \exp[-c_2(k-i)]$$

$$\triangleq g(Z_{k-1}) \text{ independent of } j\ . \tag{3.69}$$

Therefore, we have

$$\|\hat{x}(k\ k;\pi_q)\| \leq g(Z_{k-1}) \tag{3.70}$$

Now, the Cauchy-Schwartz inequality yields

$$[\hat{x}(k|k;\pi_q)\ F\ \hat{x}^T\ (k|k;\pi_r)]^2 \leq \|F\|^2 \times g^4(Z_{k-1}) \tag{3.71}$$

It can be shown that (e.g. see the proof of Lemma 3.7)

$$E\{g^4(Z_{k-1})\} < \infty$$

and hence, the desired result follows from (3.71).

Q.E.D.

The following theorem implies the convergence of $[\hat{x}(k|k)-\hat{x}(k|k;\pi_t)]$ to zero in quadratic mean.

Theorem 3.3: Suppose that the hypotheses of Theorem 3.2 and Lemma 3.9 are satisfied. Then

$$\lim_{k \to \infty} E\{[\hat{x}(k|k) - \hat{x}(k|k;\pi_t)]^T F[\hat{x}(k|k) - \hat{x}(k|k;\pi_t)]\} = 0$$

Proof: It follows from a modification in the proof of Theorem 1 in [24]. To this end, we note that, by an extension of the dominated convergence theorem (problem 5, p. 96 in [5]), $a_k \to 0$ in probability implies that $E\{a_k\} \to 0$ as $k \to \infty$, where a_k, $k = 1,2,---$ is a sequence of positive random variables such that $0 \leq a_k \leq 1$. We use this fact to appropriately modify the hypotheses of Lemma 1 and Theorem 1 in [24].

Q.E.D.

Now we show that, under the same conditions as for Theorem 3.3, the optimal quadratic risk J_k^* converges to the average optimal risk J_k^{**} for the known transition matrix estimator where

$$J_k^* = E\{[x_k - \hat{x}(k|k)]^T F[x_k - \hat{x}(k|k)]\}$$

and

$$J_k^{**} = E\{[x_k - \hat{x}(k|k;\pi_t)]^T F[x_k - \hat{x}(k|k;\pi_t)]\} .$$

Theorem 3.4: Suppose that the hypotheses of Theorem 3.2 and Lemma 3.9 are satisfied. Then

$$\lim_{k \to \infty} [J_k^* - J_k^{**}] = 0$$

Proof: It follows from a modification in the proof of Theorem 2 in [24]. The modification required is the same as given in the proof of Theorem 3.3.

Q.E.D.

We remark that the convergence theorems in this section are general and apply to any (stable) conditional mean estimation system. For example, suppose that Case 1 of model (1.1)-(1.2) is modified to include a control input (possibly closed loop) in state equation (1.1). Then if Theorem 3.2 can be shown to hold, results of this section will follow easily.

3.6. Example

In this section, a simple example is presented to illustrate the convergence of the a posteriori probability of π given the observations.

We consider a scalar linear system described by the following equations:

$$x_{k+1} = 0.9\, x_k + w_k$$

$$z_k = x_k + v_k \quad , \quad k = 0,1,\text{---}$$

$x_o \sim N(1,2)$, $v_k \sim N(0,1)$ $\forall$ k

w_k is zero-mean, conditionally white, Gaussian noise process and its covariance switches between $Q^{(1)} = 0.5$ and $Q^{(2)} = 10$. The transition matrix π is unknown. It is assumed that $\pi \in \{\pi_1, \pi_2, \pi_t\}$ where π_t represents the correct value of π and

$$\pi_t = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}, \quad \pi_1 = \begin{bmatrix} 0.995 & 0.005 \\ 0.005 & 0.995 \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

The Markov chain states belong to $\{1,2\}$. We take the initial probabilities on the states to be equal to the steady state probabilities, which in this case are

$$P(i_o = 1|\pi) = P(i_o = 2|\pi) = 0.5$$

for all the three admissible π's.

The system was simulated using a random number generator and the a posteriori probabilities for the various candidate values of π were computed for a single sample sequence of observations. Figures 3.1 and 3.2 illustrate the convergence of the a posteriori probabilities. In Fig. 3.1 results are given for the random sampling approach where the number of sequences sampled were 40. In Fig. 3.2 results are given for the pseudo-Bayes approximation. It is seen from these figures that the correct value of the transition matrix was identified in less than 20 stages and convergence was achieved in less than 180 stages.

3.7. Discussion

In this chapter we presented an adaptive Bayesian approach to optimal state estimation in linear discrete-time systems with unknown Markovian noise statistics. First, expressions for the optimal estimator were derived and expressed in a recursive form. Then several suboptimal

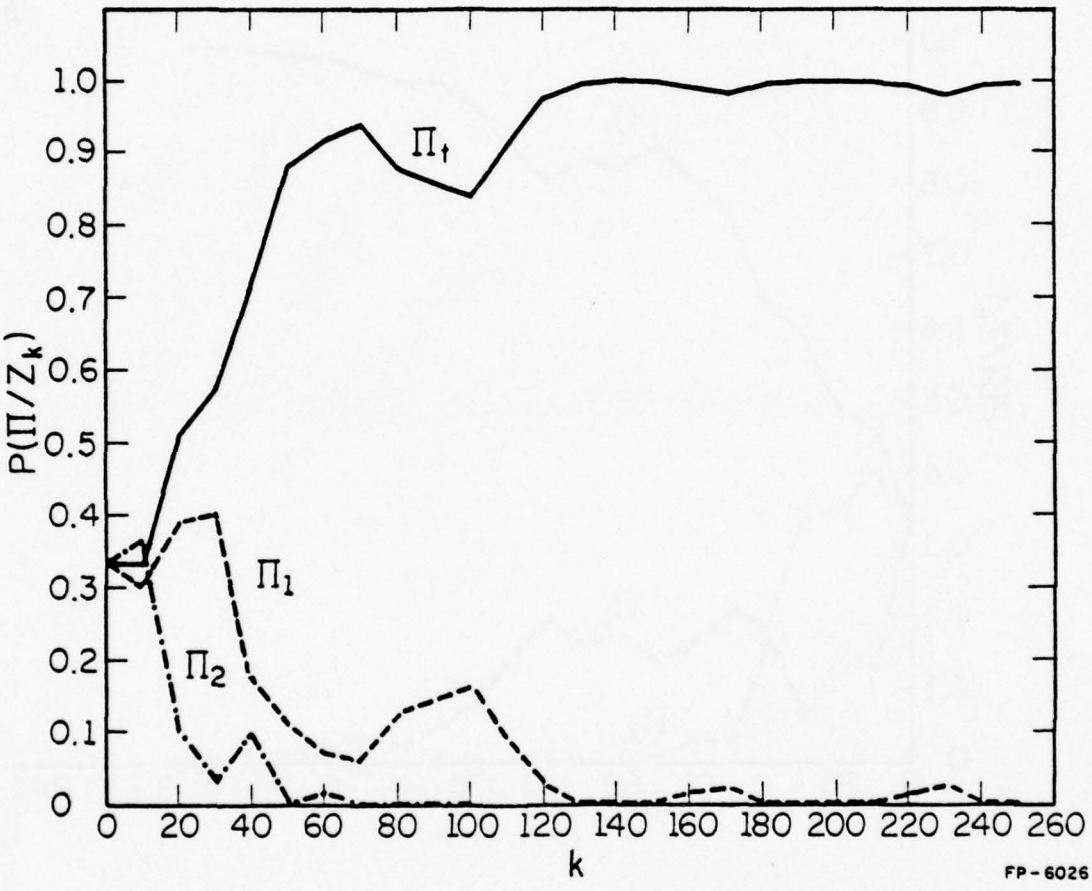

Figure 3.1. The a posteriori probabilities $P(\pi_q|Z_k)$ vs. time for the random sampling approach.

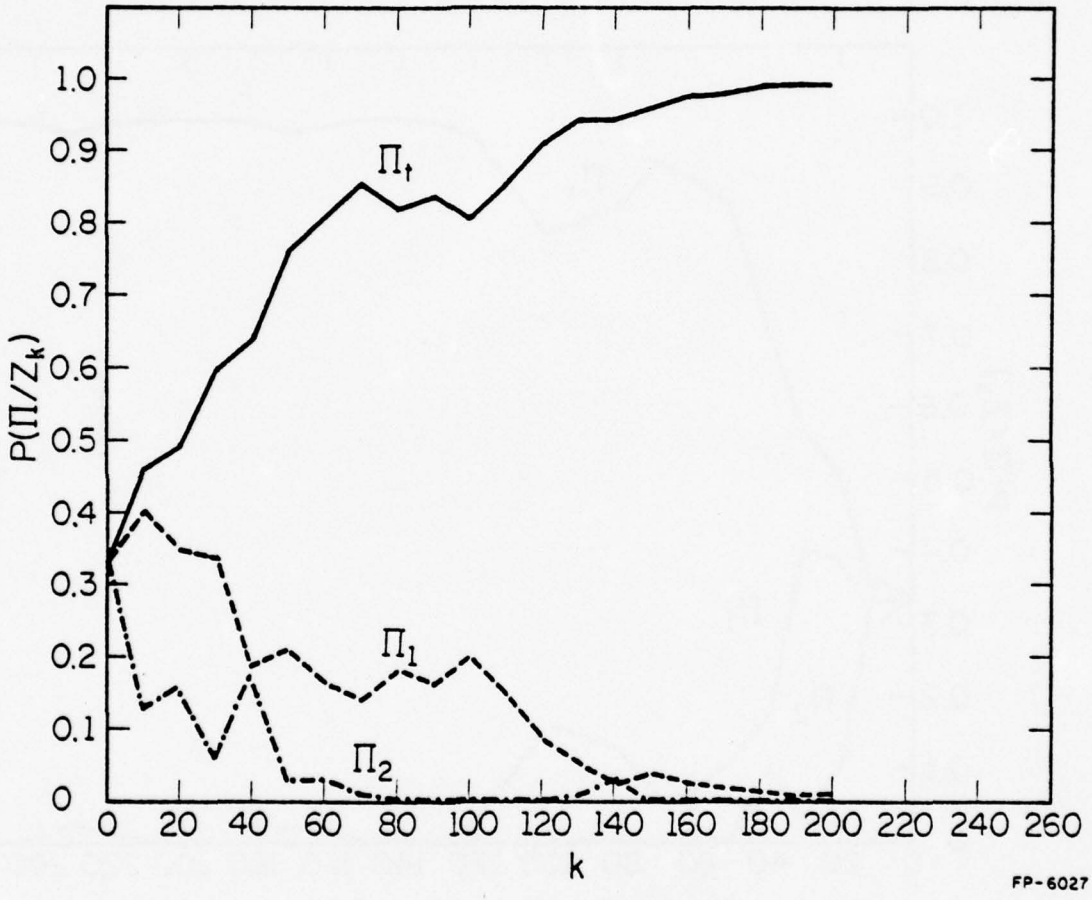

Figure 3.2. The a posteriori probabilities $P(\pi_q|Z_k)$ vs. time for the pseudo-Bayes approximation.

algorithms were discussed to alleviate the large computation and storage requirements of the optimal solution. Finally, asymptotic behavior of the solution was investigated. We examined the Bayes optimal estimation scheme, whereas, in practice, only suboptimal algorithms can be used. Assuming that the suboptimal algorithm is a "good" approximation to the optimal scheme, our convergence results should then carry over to the suboptimal schemes in some sense.

An example was simulated using random sampling approach and the pseudo-Bayes approximation. It demonstrated the convergence of the conditional probability of the true transition matrix, given the past observations, to one. It should be noted, however, that since, in practice, only suboptimal algorithms can be employed, therefore, despite Theorems 3.1 and 3.2, we may not be able to resolve two arbitrarily "close" transition matrices. This resolvability will, in general, depend upon the "goodness" of the approximation to the optimal scheme and can, perhaps, be determined only through simulations. The significance of Theorem 3.2 lies in the fact that it indicates insensitivity of the asymptotic performance to the priors $P(\pi_q)$. Moreover, the question of resolvability of the unknown parameter set is of practical interest in that it facilitates the selection of an appropriate model structure (unknown parameter set). If our primary interest lies in the asymptotic performance and if two unresolvable transition matrices are included in the set of unknown transition matrices, then by discarding one of the unresolvable matrices a saving in the computational requirements can be obtained. The sufficient conditions given in this chapter aim to answer this question.

In establishing the consistency results given in this chapter, we used the sufficient conditions given in Tse and Aton [63]. These are a generalization of the conditions given by Wald [66] for independent equidistributed observation sequence. In the literature other sets of sufficient conditions are also available; one of them is a generalization of the conditions given in Cramér [16]. These conditions have two drawbacks: one is some differentiability and uniformity conditions on the observation density which are quite difficult to verify, and the other drawback is that these conditions give only "local" results as opposed to the "global" results given by the sufficient conditions of Tse and Anton [63].

We considered only the case where the set of unknown transition matrices is finite (and discrete). When the set of unknown matrices is compact (and continuous), one way to extend the given method to this case is to approximate the continuous parameter space with a finite set of quantized points, and then solve the resulting finite parameter space approximation to the problem by the method presented here. A major difficulty with such a quantization technique is that the quantized points used in the approximation increase exponentially with the dimension of the parameter space. One possible alternative to this difficulty may be the combined detection-estimation scheme, with incremental mean-square error criterion of Sebald and Haddad [55].

CHAPTER 4

ADAPTIVE ESTIMATION UNDER UNCERTAIN OBSERVATIONS

4.1. Introduction

In Chapter 3 we considered the problem of adaptive state estimation in switching environments under the assumption that the transition probability matrix was unknown. In this chapter we are concerned with state estimation for linear discrete-time systems with uncertain observations, where the uncertainties are governed by a Markov chain with unknown transition probability matrix. Specifically, we consider Case 2 of Model (1.1)-(1.2). We have chosen to treat Case 2 separately from Case 1 because (i) it leads to considerable simplification in notation and (ii) more importantly, the sufficient conditions required to prove convergence results for Case 2 in Section 4.3 are different from the conditions needed for Case 1 (see Section 3.4).

An important aspect of the problem of state estimation for linear systems is the case where the observations do not contain the signal with probability one. As pointed out in Section 1.1, such a situation may occur due to an intermittent failure in the observation mechanism. It may also occur in a tracking situation when sensor returns may originate from something other than a single object being tracked, such as other objects being tracked, new objects not yet being tracked, false alarms, clutter, and radio frequency interference [56]. Sawaragi et. al. [51,52] were the first to address the problem of state estimation for

linear discrete-time systems with stationary interrupted observation mechanism where the statistics of the interruption process are unknown but fixed. In both these papers a Bayesian viewpoint was adopted and the objective was to present a feasible adaptive algorithm that sequentially produced an approximate MMSE estimator. The asymptotic convergence of the a posteriori probabilities of the Markov transition probabilities, given the past observations, was demonstrated through simulations; no theoretical justification was provided

Our main concern in this chapter is to provide theoretical justification for the convergence of the a posteriori probabilities of the transition probability matrices. The objective is to investigate the conditions under which the constrained maximum likelihood estimate of the transition matrix converges in probability to the correct value of the transition matrix. Then other convergence results such as convergence of the a posteriori probability of the transition matrix given the past observations and convergence of the performance of the adaptive scheme, follow easily as in Sections 3.4 and 3.5.

In Section 4.2 we discuss state estimators for Case 2 of Model (1.1)-(1.2); both MMSE estimator, optimal in the Bayesian sense, and suboptimal approximations to it are considered. The convergence results pertaining to the optimal MMSE estimator are discussed in Section 4.3.

4.2. State Estimators

In this section state estimators for Case 2 of Model (1.1)-(1.2) are discussed. We treat both, Bayes optimal MMSE estimator and suboptimal approximations to it.

Optimal Estimator: The derivation of the optimal estimator parallels that given in Section 3.2 for Case 1. Therefore, we shall omit most of the details and simply indicate the changes to be made in the equations given in Section 3.2, appropriate to Case 2.

Note that since only $\{\gamma_k\}$ is assumed to be modeled by a Markov chain, we only have a two-state Markov chain, with unknown transition probability matrix $\pi \in \{\pi_1, \pi_2, \ldots, \pi_M\}$. Let the a priori probability of the q^{th} transition matrix be denoted by $P(\pi_q)$ as in Section 3.2. The problem, as in Section 3.2, is to obtain the conditional mean of state x_k given the past observations Z_k. Define a Markov chain state sequence I(k) as

$$I(k) = \{\gamma_0, \gamma_1, \ldots, \gamma_k\} \tag{4.1}$$

which replaces Equation (2.2) in Section 2.2. Let $I_j(k)$ denote a specific state sequence from the space of the sequences I(k) which contains 2^{k+1} elements. Furthermore, $I_j(k)$, $I_\ell(k-1)$, m and n used in Equation (3.7) are now defined by the relations

$$I_j(k) = \{I_\ell(k-1),\ \gamma_k = m\}, \quad m = 0 \text{ or } 1 \tag{4.2}$$

and

$$I_\ell(k-1) = \{\gamma_0, \gamma_1, \ldots, \gamma_{k-2}, \gamma_{k-1} = n\}, \quad n = 0 \text{ or } 1 \tag{4.3}$$

All other equations in Section 3.2 remain the same except that the Kalman filter equations needed to compute $f(z_k|Z_{k-1}, I_j(k))$ and $\hat{x}_j(k|k)$ are now appropriately modified to suit Case 2 of Model (1.1)-(1.2). These modifications will be discussed later in Section 4.3 (see Equations (4.4)-(4.9)).

As pointed out in Section 2.2 the optimal estimator requires exponentially increasing storage and computation capabilities. To circumvent this difficulty, we now discuss several suboptimal approximations.

Suboptimal Estimators: All the three approximations discussed in Section 3.3 are applicable here, with minor changes caused by switchings in the observation matrix C_k (see Equation (1.2)). The changes are straightforward and therefore, will not be discussed any further.

In the following section we discuss some convergence results pertaining to the optimal MMSE estimator.

4.3. Convergence

In this section the asymptotic behavior of the Bayes optimal MMSE estimator is investigated. Attention is focused on investigation of the sufficient conditions under which the CML (constrained maximum likelihood) estimate of the transition probability matrix converges in probability to the correct value of the transition matrix. Then other convergence results such as convergence of the a posteriori probability of the transition matrix given the past observations and convergence of the performance of the adaptive scheme, follow easily as in Sections 3.4 and 3.5.

It should be noted that in the sequel, as in Chapter 3, the distinction between a random vector and its particular realization (value) will not always be made in the notation, rather it will be left to context

The Kalman filter matched to a specific state sequence $I_j(k)$ is given by the following equations:

$$\hat{x}_j(k|k) = \Psi_j(k,k-1)\hat{x}_j(k-1|k-1) + K_j(k)z_k \tag{4.4}$$

$$K_j(k) = P_x(k|k-1;j|j)\gamma_{k_j} C^T[CP_x(k|k-1;j|j)C^T\gamma_{k_j}^2 + R]^{-1} \tag{4.5}$$

$$P_x(k|k;j|j) = \Psi_j(k,k-1)P_x(k-1|k-1;j|j)\Psi_j^T(k,k-1) + F_j(k) \tag{4.6}$$

$$F_j(k) = [I-K_j(k)C\gamma_{k_j}]BQB^T[I-K_j(k)C\gamma_{k_j}]^T + K_j(k)RK_j^T(k) \tag{4.7}$$

where

$$\Psi_j(k,k-1) = (I-K_j(k)C\gamma_{k_j})A \tag{4.8}$$

$$\Psi_j(k,\ell) = \Psi_j(k,k-1)\Psi_j(k-1,k-2)\cdots\Psi_j(\ell+1,\ell) \tag{4.9}$$

$$P_x(k|k-1;j|j) = E\{[x_k - \hat{x}_j(k|k-1)][x_k - \hat{x}_j(k|k-1)]^T|I_j(k)\}$$

and

γ_{k_j} is the "last" element in the sequence $I_j(k)$.

We impose the following conditions on Case 2 of Model (1.1)-(1.2):

(CH 1) The state transition matrix A in the system Model (1.1)-(1.2) has all its eigenvalues inside the unit circle.

(CH 2) Assume that, for some $\delta > 0$,

$$\max_{\lambda} |\lambda([I-K_j(k)C\gamma_{k_j}]A)|^{\ddagger} \leq 1 - \delta, \ \forall \ k \text{ and } j, \quad j = 1,2,\ldots,2^{k+1},$$

where $K_j(k)$ and γ_{k_j} are as defined in the Kalman filter Equations (4.4)-(4.9). Furthermore, assume that the pair (A,C) in (1.1)-(1.2) is observable.

(CH 3) For all $\pi \in \Omega$, we have $p_{00}, p_{11} < 1$, where $\Omega = \{\pi_1,\pi_2,\ldots,\pi_M\}$. A consequence of this is that the 2-state Markov chain governed by any $\pi \in \Omega$ is ergodic [19] (see also Remark 3.1 in Chapter 3).

(CH 4) Suppose that the system (1.1)-(1.2) begins at initial time $k = k_0$. Let $I_j(n,k)$ represent a specific sequence from the set of Markov chain state sequences $I(n,k) \triangleq \{\gamma_k,\gamma_{k+1},\ldots,\gamma_n\}$, $n \geq k$, $j = 1,2,\ldots,2^{n-k+1}$. Let $Z_{n,k_0} = \{z_k, k_0 \leq k \leq n\}$. Assume the following:

$$\lim_{k_0 \to -\infty} P(I_j(n,k)|\pi_q,Z_{n,k_0}) = P(I_j(n,k)|\pi_q,Z_{n,-\infty}) \quad \text{a.e. } P^Z_{\pi_t}$$

where $n,k > -\infty$ are fixed, and $P^Z_{\pi_t}$ is a probability measure induced on the range space of the random sequence $\{---,z_k,---,z_n\}$ parameterized by the correct value of the transition matrix π.[†]

The following theorem states the weak consistency of the CML estimate of π.

[†]Note that here we are using the same notation for a random variable and its values.

[‡]$\lambda(B) \triangleq$ an eigenvalue of matrix B.

Theorem 4.1: Suppose that the conditions (CH 1)-(CH 4) hold. Then the CML estimate $\hat{\pi}(Z_n)$ of π converges in probability to π_t as $n \to \infty$.

The proof for Theorem 4.1 consists of satisfying the sufficient conditions given in Tse and Anton [63], as in the case of Theorem 3.1 in Chapter 3. Several preliminary results are needed first, some of which parallel those given in Chapter 3.

Lemma 4.1: Suppose that conditions (CH 1)-(CH 2) are satisfied. Then the homogeneous part of the Kalman filter matched to any Markov chain state sequence is uniformly asymptotically stable, uniformly in time and in all possible state sequences.

Proof: The homogeneous part of the Kalman filter matched to $I_j(k)$ is given by

$$\hat{x}_j(k|k) = \Psi_j(k,k-1)\hat{x}_j(k-1|k-1) \tag{4.10}$$

From condition (CH 2) and Equation (4.8) it follows that $\Psi_j(k,k-1)$ is uniformly asymptotically stable, uniformly in j and in k. Therefore, there exist real numbers $c_1, c_2 > 0$ (independent of k and j) such that [30]

$$\|\Psi_j(k,\ell)\| \leq c_1 \exp[-c_2(k-\ell)] \quad \forall\ j \tag{4.11}$$

Q.E.D.

The condition (CH 2) is crucial to the proof of Lemma 4.1, however, alternate conditions may be used to replace (CH 2). These assumptions will be denoted by (CH 2.i), i = 1,2,3.

Assumption (CH 2.1): Assume that

$$P_x(k|k-1;j|j)C^T R^{-1} C \geq 0 \qquad \forall\ k$$

Furthermore, the pair (A,C) is observable.

Together with condition (CH 1), condition (CH 2.1) can then be used to prove Lemma 4.1. To show this, from [28, Chap. 7, Equation (7B.1)], we have

$$I - K_j(k)C\gamma_{k_j} = [I + \gamma_{k_j}^2 P_x(k|k-1;j|j)C^T R^{-1} C]^{-1} \tag{4.12}$$

Therefore, from Equation (4.8) it follows that

$$\|\Psi_j(k,k-1)\| \leq \|[I + \gamma_{k_j}^2 P_x(k|k-1;j|j)C^T R^{-1} C]^{-1}\| \cdot \|A\|^{\dagger} \tag{4.13}$$

Now condition (CH 2.1) yields

$$\|[I + \gamma_{k_j}^2 P_x(k|k-1;j|j)C^T R^{-1} C]^{-1}\| \leq \|I\| = 1 \tag{4.14}$$

Since A is asymptotically stable, there exist real numbers $c_1, c_2 > 0$ (independent of k and j) such that [30]

$$\|A\|^k \leq c_1 \exp[-c_2 k], \quad k \geq 0 \tag{4.15}$$

From Equations (4.9) and (4.13)-(4.15), we have

$$\|\Psi_j(k,\ell)\| \leq \|A\|^{k-\ell} \leq c_1 \exp[-c_2(k-\ell)], \quad \forall\ j \tag{4.16}$$

---

†In this thesis, we take $\|A\| = \max_{\lambda}[\lambda(A^T A)]^{\frac{1}{2}}$.

The requirement that the matrix $P_x(k|k-1;j|j)C^T R^{-1}C$ be positive semi-definite is more restrictive than (CH 2), but is easier to verify. It is obviously satisfied for a scalar dynamical system. Also consider the case where all the state variables are observed separately and independently, i.e., let C be a diagonal matrix. Then (CH 2.1) holds. Another example where (CH 2.1) holds is the case of scalar observations with C as a row vector with all entries equal to one. In general, the matrix $P_x(k|k-1;j|j)C^T R^{-1}C$ is positive semi-definite if the symmetric positive semi-definite matrices $P_x(k|k-1;j|j)$ and $C^T R^{-1}C$ commute. From a result in Bellman [8, p. 56], it follows that if two positive semi-definite matrices G and H commute, then there exists an orthogonal matrix which will simultaneously diagonalize G and H. Let W denote such an orthogonal matrix. Then $W^T W = I$, and $W^T GW$ and $W^T HW$ are diagonal matrices. Furthermore, since $G,H \geq 0$, it follows that $W^T GW$ and $W^T HW$ have only nonnegative diagonal elements; hence, $GH = W(W^T GW)(W^T HW)W^T$ is positive semi-definite.

Assumption (CH 2.2): The pair (A,C) is observable and the state equation (1.1) is controllable. Furthermore, $\|A\| < \lambda_{min}/\lambda_{max}$ where

$$\lambda_{min} \leq \inf_{\lambda} |\lambda(P_x(k|k-1;j|j))| \quad \forall\ j \text{ and } k$$

$$\lambda_{max} \geq \sup_{\lambda} |\lambda(P_x(k|k-1;j\lambda j))| \quad \forall\ j \text{ and } k$$

Now we shall prove Lemma 4.1 using Assumption (CH 2.2).

Let $I_{j_1}(k)$ denote the Markov chain state sequence with all entries equal to 1, i.e., for which $\gamma_k = 1 \ \forall \ k$. Then we have

$$P_x(k|k\text{-}1;j_1|j_1) \leq P_x(k|k\text{-}1;j|j) \quad \forall \ j \tag{4.17}$$

$j = 1,2,\ldots,2^{k+1}$, since the filter matched to $I_{j_1}(k)$ makes use of all the observations. Now from the controllability and observability assumptions and Lemma 7.2 in [28] (see also [14] and [25]), it follows that there exist $\alpha > 0$ and integer $N > 0$ such that

$$P_x(k|k\text{-}1;j_1|j_1) \geq \alpha \ I \quad \forall \ k \geq N \tag{4.18}$$

Furthermore, by using condition (CH 1) we can show that (as in Lemma 3.2) there exists $\beta > 0$ such that

$$P_x(k|k\text{-}1;j|j) \leq \beta I \quad \forall \ j \text{ and } k \tag{4.19}$$

From (4.17) and (4.18), $\alpha \leq \inf_{\lambda} |\lambda(P_x(k|k\text{-}1;j|j))|$ and from (4.19), $\beta \geq \sup_{\lambda} |\lambda(P_x(k|k\text{-}1;j|j))|$ for all j and k. Now Equation (4.8) may be rewritten as

$$\Psi_j(k,k\text{-}1) = P_x(k|k;j|j)P_x^{-1}(k|k\text{-}1;j|j)A$$

where we used Equation (7.105) in Chapter 7 of [28]. Furthermore, we have

$$P_x(k|k;j|j) = [P_x^{-1}(k|k\text{-}1;j|j) + \gamma_{k_j}^2 C^T R^{-1} C]^{-1}$$

$$\leq P_x(k|k\text{-}1;j|j) \text{ since } R^{-1} > 0 \ .$$

Hence

$$\|\Psi_j(k,k-1)\| \leq \|P_x(k|k-1;j|j)\|_x \|P_x^{-1}(k|k-1;j|j)\|_x \|A\|$$

$$\leq \lambda_{max} \times (\lambda_{min})^{-1} \times \|A\|^{\dagger}$$

$$< 1, \text{ by assumption.}$$

Therefore there exist $c_1, c_2 > 0$ such that

$$\|\Psi_j(k,\ell)\| \leq c_1 \exp[-c_2(k-\ell)], \quad \forall\ j.$$

The existence of λ_{max} and λ_{min} has already been proved. For example, take $\lambda_{max} = \beta$ and $\lambda_{min} = \alpha$. An implication of the condition (CH 2.2) is that the eigenvalues of A should be sufficiently "close" to the origin.

Assumption (CH 2.3): In model (1.1)-(1.2), Case 2, $BQB^T > 0$. Also, the observation matrix C is of full rank.

The assumption (CH 2.3) implies that there exist real numbers α_1 and β_1 such that the conditions given by (4.20) and (4.21) hold:

$$BQB^T \geq \alpha_1 I, \quad \alpha_1 > 0 \tag{4.20}$$

$$C^T R^{-1} C \geq \beta_1 I, \quad \beta_1 > 0 \tag{4.21}$$

Furthermore, there also exist real numbers α_2 and β_2 such that (4.22) and (4.23) hold:

$$BQB^T \leq \alpha_2 I, \quad \alpha_2 < \infty \tag{4.22}$$

$$C^T R^{-1} C \leq \beta_2 I, \quad \beta_2 < \infty \tag{4.23}$$

---

†For a symmetric matrix D, $\|D\| = \max_{\lambda} |\lambda(D)|$.

The conditions (4.20)-(4.23) are much stronger than the usual controllability and observability assumptions. These conditions may now be used to prove the uniform asymptotic stability of $\Psi_j(k,\ell)$, given by Equation (4.9), for all j, k, ℓ. To this end, note that, from Equation (4.8), whenever $\gamma_{k_j} = 0$, $\Psi_j(k,k-1) = A$ where A is stable by condition (CH 1). So we need to consider only those components of $\Psi_j(k,\ell)$ in (4.9) for which $\gamma_{i_j} = 1$, $k \leq i \leq \ell$. Now we may use the Lyapunov function technique, as in the proof of Lemma 3.5, to prove the asymptotic stability of $\Psi_j(k,\ell)$. We make an extensive use of the conditions (4.20)-(4.23); details are omitted.

Finally, it is conjectured that the stability of matrix A should be sufficient to ensure uniform stability of the Kalman filters matched to all possible state sequences. A proof is not yet available.

Lemma 4.2: There exist real numbers $\beta_1, \beta_2 > 0$ such that

$$\beta_1 I \leq P_z(k|k-1;j|j) \leq \beta_2 I \qquad \forall\ j,k; \quad j = 1,2,\ldots,2^{k+1}$$

where

$$P_z(k|k-1;j|j) = E\{[z_k - \hat{z}(k|k-1;j)][z_k - \hat{z}(k|k-1;j)]^T | I_j(k)\}$$

and

$$\hat{z}(k|k-1;j) = E\{z_k | Z_{k-1}, I_j(k)\}$$

$$= \gamma_{k_j} C\, E\{x_k | Z_{k-1}, I_j(k)\}$$

Proof: We have

$$P_z(k|k\text{-}1;j|j) = \gamma^2_{k_j} C\, P_x(k|k\text{-}1;j|j)C^T + R \qquad (4.24)$$

$$\geq R \qquad \forall\ k,j,\ell$$

$$\geq \beta_1 I \text{ since } R > 0 \text{ by assumption.}$$

From the condition (CH 1), it follows that there exists a positive number $M < \infty$ such that

$$\|P_x(k|k\text{-}1;j|j)\| < M \qquad \forall\ j,k \qquad (4.25)$$

(For example, take $\hat{x}_j(k|k) = \bar{x}_0 = \hat{x}_j(0|0)$ $\forall\ k,j$; a suboptimal estimator.) Hence, there exists a real number $\beta_2 > 0$ such that

$$P_z(k|k\text{-}1;j|j) \leq \beta_2 I \qquad \forall\ j,k$$

Q.E.D.

Lemma 4.3: Given the dynamical system (1.1)-(1.2), the observations z_k and z_n become independent as $|k\text{-}n| \to \infty$ for all $k,n \geq 0$.

Sketch of Proof: It follows from condition (CH 1), normality of x_0 and ergodicity of the Markov chain corresponding to the true transition matrix π_t.

Q.E.D.

Lemma 4.4: Let the dynamical system (1.1)-(1.2) begin at initial time $k_0 = -\infty$ (instead of $k_0 = 0$). Then, for $k \geq k_1 > -\infty$, the observation sequence $\{z_k\}$ is stationary.

Sketch of Proof: It follows from conditions (CH 1) and (CH 3).

Q.E.D.

Lemma 4.5: (i) Let the dynamical system (1.1)-(1.2) begin at initial time $k_0 = \ell$. Then

$$\lim_{\ell \to -\infty} f_{z_k|Z_{k-1,\ell}}(\alpha_k|\alpha_{k-1,\ell},\pi) = f_{z_k|Z_{k-1,-\infty}}(\alpha_k|\alpha_{k-1,-\infty},\pi) \quad \text{a.e. } P^Z_{\pi_t}$$

where $f_{z_k|Z_{k-1,\ell}}(\cdot|\cdot,\pi)$ is the probability density function of z_k given $Z_{k-1,\ell} = \{z_n\}_{n=\ell}$ assuming that the transition matrix π is true; $\alpha_{k,\ell} = \{\alpha_i\}_{i=\ell}^{k}$ denotes the value taken by the random sequence $\{z_i\}_{i=\ell}^{k}$.
(ii) Let the dynamical system (1.1)-(1.2) begin at initial time $k_0 = -\infty$. Then the random sequence $\{f_{z_k|Z_{k-1}}(z_k|Z_{k-1},\pi\}_{k=0}^{\infty}$ becomes asymptotically stationary, where $Z_{k-1} = \{z_n, 0 \le n \le k-1\}$.

Proof: It parallels entirely the proof of Lemma 3.6.

Lemma 4.6: The random variable $\ell n\, f_{z_k|Z_{k-1,-\infty}}(z_k|Z_{k-1,-\infty},\pi)$ is integrable, $k = 0,1,\ldots$, where $Z_{k,-\infty} = \{z_i, -\infty \le i \le k\}$.

Proof: It is the same as the proof of Lemma 3.7.

Lemma 4.7: Define

$$y_k = \ell n\, f(z_k|Z_{k-1},\pi) - E\{\ell n\, f(z_k|Z_{k-1},\pi)\}$$

where

$$Z_k = \{z_i, 0 \le i \le k\}. \quad \text{Then}$$

$$\Pr\{n^{-1}|y_1 + y_2 + \ldots + y_n| > \varepsilon\} \to 0 \text{ as } n \to \infty$$

for any $\varepsilon > 0$.

Proof: From Lemmas 4.3 and 4.5, it follows that $E\{y_\ell y_k\} \to 0$ as $|k-\ell| \to \infty$. Then use the result in Feller [20, p. 240] and asymptotic stationarity of the sequence $\{y_k, 0 \le k \le \infty\}$ to obtain the desired result.

Q.E.D.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1: It consists of satisfying the 6 conditions of Tse and Anton [63], discussed in the proof of Theorem 3.1 in Chapter 3. Conditions (AC1) through (AC4) are satisfied, as in the proof of Theorem 3.1, using the results of Lemmas 4.1 through 4.7. To show that the condition (AC5) is satisfied, we note that, for $\pi \neq \pi_t$, we have $\lim_{n \to \infty} \pi^n \neq \lim_{n \to \infty} (\pi_t)^n$, consequently, the observation process $\{z_k\}$ corresponding to these π's become, in the limit, two stationary processes with different statistics (see Lemma 4.4). Then $\lim_{n \to \infty} f(z_n|Z_{n-1},\pi) \neq \lim_{n \to \infty} f(z_n|Z_{n-1},\pi_t)$ over a set of nonzero probability. Combining this with Lemma 4.5, it follows that condition (AC5) is satisfied Then condition (AC6) can be shown to hold as in the proof of Theorem 3.1. This proves Theorem 4.1.

Q.E.D.

Remark 4.1: It concerns condition (CH 4). The comments made in Remark 3.7 are applicable here too.

By using Theorem 4.1, we can easily establish convergence results similar to those given in Theorems 3.2, 3.3 and 3.4. We omit the details to avoid unnecessary repetitions. Finally, it should be noted that the scheme outlined in Section 4.2 is optimum in the Bayes sense at every step k and the optimality, therefore, is independent of convergence. Nevertheless, it is of interest to know the conditions under which the solution converges to one that uses the correct transition probabilities, since it makes the asymptotic solution independent of the priors $P(\pi_q)$.

4.4. Discussion

The asymptotic behavior of a Bayes optimal adaptive estimation scheme for a linear, discrete-time system with Markov interrupted observations was investigated, where the transition probability matrix of the Markov chain governing the interruption process was assumed to be unknown. The main objective was to provide theoretical justification for some simulation results in [52]. We examined the Bayes optimal estimation scheme whereas in [52] simulation results were given using a suboptimal algorithm. Assuming that the suboptimal algorithm is a "good" approximation to the optimal scheme, our convergence results should then carry over to the suboptimal scheme in some sense.

The convergence results given in this paper can easily be extended to a slightly more general dynamical model where the measurement Equation (1.2) is replaced by

$$z_k = \gamma_k C x_k + (1-\gamma_k) v_k^{(1)} + v_k^{(2)}$$

where $\{v_k^{(i)}\}$, $i = 1,2$, are two independent, zero-mean, white Gaussian noise sequences with constant covariances (see e.g. [37]).

Finally, comments made in Section 3.7 apply here too.

CHAPTER 5

ESTIMATION FOR SYSTEMS WITH UNKNOWN, TIME-INVARIANT NOISE COVARIANCES

5.1. Introduction

So far, in the last few chapters, we have dealt only with a particular type of parameterization/modelling of the switching characteristics of a linear discrete-time system. Specifically, we assumed that a finite-state Markov chain describes the switchings in the noise characteristics. Such a probabilistic description may not always be appropriate. In this chapter, and also the next, we investigate a detection-estimation scheme for state estimation in uncertain dynamical systems with unknown noise covariances. Information regarding the unknown parameters is assumed to be available in the form of multiple bounds on the values of the parameters and their time-derivatives. In this chapter attention is confined to constant noise covariances; the time-varying case is treated in the next chapter.

The proposed scheme is based, to an extent, on the approach given in [46]. It is a heuristic extension of the standard minimax scheme to the case when multiple bounds (disjoint or nested) on the unknown parameters are available. To this end, a weighted conditional mean-squared error cost structure is formulated. An ad-hoc solution which draws upon the properties of the standard minimax solution to the single bounding set case, is proposed. The proposed approach is an attempt to alleviate the pessimistic performance of the standard minimax estimator for large observation records and large uncertainties, while retaining its desirable small-sample properties. A Bayesian approach [34], [36] would require

specification of the a priori probabilities of occurrence of all possible parametric values. If the "true" a priori probabilities are "much different" from the assumed probabilities, small-sample performance will suffer. The small-sample performance of a given scheme is particularly important for time-varying parameter case when not much is known regarding the time-variation of the parameter. The given scheme is expected to be useful mainly in the time-varying parameter case; the present work limited to constant parameters is intended as an illustration of the concepts involved.

The proposed scheme is formulated in Section 5.2. A weighted conditional mean-square error cost structure is proposed in Section 5.2, to which an approximate solution is given in Section 5.3. In Section 5.4 the asymptotic behavior of the solution is investigated. In Section 5.5 an example is considered to illustrate the results.

5.2. Proposed Scheme

We consider Case 3 of model (1.1)-(1.2). Recall that the given model is specified up to a set of time-invariant unknown parameters denoted by the vector θ of dimension q. This uncertainty may be in constant noise covariances Q or R only, which are assumed to be continuous functions of θ. The unknown parameter vector θ is assumed to satisfy at least one of the following conditions

$$\theta \in \Omega_i, \quad i = 1,2,\ldots,N$$

where Ω_i, $i = 1,2,\ldots,N$ is a collection of compact subsets of $\mathbb{R}^q$.

It is desired to estimate the value of x_k based on the observations $Z_k = \{z_j,\ 0 \le j \le k\}$.

Let $\hat{x}(k)$ denote an estimate of x_k based on the observation Z_k. The conditional mean-squared error (CMSE) of this estimate given the observation Z_k and the unknown parameter θ is given by

$$L[\hat{x}(k),Z_k,\theta] = E\{\|x_k - \hat{x}(k)\|^2 | Z_k,\theta\}.$$

It is required that the selected estimator minimize

$$\max_{\theta \in \Omega_i} L[\hat{x}(k),Z_k,\theta], \quad i=1,2,\ldots,N \tag{5.1}$$

This is a multiple objective optimization problem. It is generally not possible to minimize (5.1) with respect to $\hat{x}(k)$ for all i; the minimization of the maximum error for values of the parameter in different regions might be conflicting. So we need a scalar cost functional. The optimal estimator is defined as the one that minimizes a weighted sum of (modified) maximum errors given as

$$\sum_{i=1}^{N} \max_{\theta_i \in \Omega_i} \{\lambda_i' L[\hat{x}(k),Z_k,\theta_i]\} \tag{5.2}$$

where

$$\lambda_i' = \lambda_i f(Z_k|\theta_i)/(\sum_{j=1}^{N} \lambda_j f(Z_k|\theta_j))^* \tag{5.3}$$

and

$$\sum_{i=1}^{N} \lambda_i = 1, \ \lambda_i > 0, \ i=1,2,\ldots,N$$

Here θ_j implies $\theta \in \Omega_j$. In the Expression (5.2) the maximizing θ_i is selected while keeping θ_j, $j \neq i$, $j=1,2,\ldots,N$, fixed. (Note that $L[.,.,.]$ is also a function of θ_j, $j \neq i$, $j=1,2,\ldots,N$, through $\hat{x}(k)$.) A consistent solution to (5.2) for maximizing θ_i, $i=1,2,\ldots,N$ may not exist. We assume that either it does, or an order of precedence among θ_i's, for maximization, is specified.

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*$f(Z_k|\theta_i)$ denotes the density function of Z_k given θ_i. In Section 5.4, to avoid confusion, we also use the notation $f(Z_k;\theta_i)$ for $f(Z_k|\theta_i)$.

The weighting parameters λ_i' measure the relative importance attached to errors in each region Ω_i. The fixed parameters λ_i are design parameters that measure the a priori imporatnce assigned to the maximum errors in each region. The other part of λ_i' is observation dependent. Use of the relative conditional probability of occurrence of the given observation in the weighting parameters ensures that the optimum estimator does not minimize the maximum error in a particular region at the expense of too large an increase in the maximum CMSE in other regions, when the relative probability of occurrence of the given observation due to θ in the former region is small.

The scalar cost functional (5.2) can be viewed as a generalization of the performance criterion of Magill [36]. If the domain of θ is finely quantized such that the Ω_i's are disjoint sets each with a single element corresponding to a possible value of θ, and λ_i represents the a priori probability that $\theta \in \Omega_i$, then the given scalar cost functional reduces to the performance criterion of Magill; because then $\lambda_i' = P\{\theta \in \Omega_i | Z_k\}$.

If the objective is to minimize the mean-squared error given θ in some minimax sense, the proposed criterion is clearly more pessimistic. From a game-theoretic viewpoint, in the proposed criterion, nature chooses the worst possible a posteriori strategy whereas in case of the usual CMSE cost function nature chooses the worst a priori strategy. In other words, nature is allowed the knowledge of estimator's observation in addition to what she knows in case of the usual CMSE cost functional.

Now the problem is as given in (5.4),

$$\min_{\hat{x}(k) \in S} \sum_{i=1}^{N} \max_{\theta_i \in \Omega_i} \{\lambda_i' L[\hat{x}(k), Z_k, \theta_i]\} \tag{5.4}$$

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where S denotes the space of all measurable transformations from the observation space Z_k into $\mathbb{R}^n$. Exact solution to (5.4) appears to be intractable. An ad-hoc solution is investigated. The structure of the estimator is assumed to be of the form given by (5.5).

$$\hat{x}(k) = \sum_{i=1}^{N} \lambda_i' \hat{x}_i(k) \tag{5.5}$$

where $\hat{x}_i(k)$ denotes the MMSE estimator matched to $\theta_i \in \Omega_i$ and based on observation Z_k. Note that we are also assuming that θ_i's in (5.5) are the same as the worst case parameter values in Ω_i's. This structure is similar to the one resulting from a Bayesian viewpoint [34], [36]. Also, when there is just one bounding set (i.e., N = 1), we obtain the standard minimax solution. Now, this amounts to solving for a maximin solution to (5.4). For this to be a valid solution, we must show that problem (5.4) when viewed as a game, has a value and nonrandomized strategies exist for both $\hat{x}(k)$ and θ_i, i = 1,2,...,N. That is, the following must hold,

$$\min_{\hat{x}(k)\in S} \sum_{i=1}^{N} \max_{\theta_i \in \Omega_i} \{\lambda_i' L[\hat{x}(k), Z_k, \theta_i]\} = \sum_{i=1}^{N} \max_{\theta_i \in \Omega_i} \{\lambda_i' L[\hat{x}_0(k), Z_k, \theta_i]\}$$

where

$$\hat{x}_0(k) = \arg\{ \min_{\hat{x}(k)\in S} \sum_{i=1}^{N} \lambda_i' L[\hat{x}(k), Z_k, \theta_i]\} .$$

So far we have been unable to show this. Now, in general, a maximin solution is very pessimistic. However, it is well known [17], [47] that

for a standard minimax scheme with quadratic cost function, the game has a value when uncertainty lies only in noise covariances. Heuristically extending this approach to the multiple bounding sets case, we would like to investigate the performance of the estimator given by (5.5).

The problem now is that of finding θ_i to maximize

$$L_i \triangleq \lambda_i' E\{\|x_k - \hat{x}(k)\|^2 | Z_k, \theta_i\} \tag{5.6}$$

$$= \lambda_i' E\{\|x_k - \hat{x}_i(k)\|^2 | Z_k, \theta_i\} + \lambda_i' \|\hat{x}(k) - \hat{x}_i(k)\|^2 . \tag{5.7}$$

Substituting (5.5) in (5.7) and noting that $\sum_{i=1}^{N} \lambda_i' = 1$, we have

$$L_i = \lambda_i' E\{\|x_k - \hat{x}_i(k)\|^2 | Z_k, \theta_i\} + \lambda_i' \Big\| \sum_{\substack{j=1 \\ j \neq i}}^{N} \lambda_j' (\hat{x}_j(k) - \hat{x}_i(k)) \Big\|^2 \tag{5.8}$$

It can be shown [28, Chap. 7] that for linear models

$$E\{\|x_k - \hat{x}_i(k)\|^2 | Z_k, \theta_i\} = E\{\|x_k - \hat{x}_i(k)\|^2 | \theta_i\} \tag{5.9}$$

Therefore, when θ_i, $i = 1,2,\ldots,N$ are known, L_i can be generated sequentially as a function of time using N Kalman filters one each matched to θ_i.

Now we have to find

$$\arg\{ \max_{\theta_i \in \Omega_i} L_i \}, \quad i = 1,2,\ldots,N .$$

In general no closed-form analytical solution exists. In the next section a general procedure is given to find a numerical solution.

5.3. Approximate Solution

As mentioned in the last section no closed-form analytical solution exists hence an approximate numerical solution is proposed. It is obtained by parameterizing the L_i in (5.8) by approximating the continuous parameter space with a finite set of quantized points. Let Ω_i be quantized into $\Omega_i = \{\theta_{i\ell}, \ell = 1,2,\ldots,m_i\}$, $i = 1,2,\ldots,N$. Then we have a number of Kalman filters one each matched to $\theta_{i\ell}$, $\ell = 1,2,\ldots,m_i$; $i=1,2,\ldots,N$. This gives us $\hat{x}_{i\ell}(k)$ for each possible i,ℓ, where the subscript $i\ell$ on $\hat{x}$ denotes a MMSE estimator matched to parameter $\theta_{i\ell}$. As a byproduct we obtain $E\{\|x_k - \hat{x}_{i\ell}(k)\|^2 | Z_k, \theta_{i\ell}\}$ and the probabilities $f(Z_k|\theta_{i\ell})$ for all i and ℓ. All these computations can be performed recursively (see, for example, [57]). The Kalman filters are run in parallel as in [34], [36].

Now L_i can be found for all possible θ_i's. For every k find

$$\theta_{i0} = \arg\{\max_{\theta_i \in \Omega_i} L_i\}, \quad i = 1,2,\ldots,N \tag{5.10}$$

Note that here we are solving N coupled expressions. Substituting these θ_{i0}, $i = 1,2,\ldots,N$ in (5.5) gives the required estimator $\hat{x}(k)$.

Note that the memory requirements are constant with time. However, the number of quantizations required may be large, particularly if the dimension of the unknown parameter θ is large. This tends to make the proposed scheme computationally involved for large dimensional θ. Note that the computational complexity of the given scheme is of the same order as that of some of the adaptive schemes [34], [36]. Further, such

a parallel filtering structure has also been proposed in various other schemes [35], [54], [58]. The problem at hand is basically a nonlinear estimation problem. As is generally the case, a "global" solution [58] to such a problem is computationally complex.

5.4. Asymptotic Behavior

In this section we show that, under certain conditions, the solution obtained in Section 5.2 converges to a MMSE estimate matched to a value of the unknown parameter θ which belongs to the same bounding set Ω_i as does the true parameter value θ_0. The results are given only for the case where θ may belong to one of two bounding sets Ω_1 and Ω_2. Extension to larger number of sets Ω_i, $i \geq 3$ is straightforward; hence, it is omitted.

Equation (5.8) has the following form for the case of 2 bounding sets:

$$L_1 = \lambda_1' \, E\{\|x_k - \hat{x}_1(k)\|^2 | \theta_1\} + \lambda_1'(\lambda_2')^2 \|\hat{x}_2(k) - \hat{x}_1(k)\|^2 \tag{5.11}$$

$$L_2 = \lambda_2' \, E\{\|x_k - \hat{x}_2(k)\|^2 | \theta_2\} + \lambda_2'(\lambda_1')^2 \|\hat{x}_2(k) - \hat{x}_1(k)\|^2 \tag{5.12}$$

where

$$\lambda_1' = \lambda_1 f(Z_k|\theta_1)/(\lambda_1 f(Z_k|\theta_1) + \lambda_2 f(Z_k|\theta_2)) \tag{5.13}$$

$$\lambda_2' = \lambda_2 f(Z_k|\theta_2)/(\lambda_1 f(Z_k|\theta_1) + \lambda_2 f(Z_k|\theta_2)) \tag{5.14}$$

$$\lambda_1 + \lambda_2 = 1, \; \lambda_1' + \lambda_2' = 1$$

The estimate in Equation (5.5) becomes

$$\hat{x}(k) = \lambda_1' \hat{x}_1(k) + \lambda_2' \hat{x}_2(k) \tag{5.15}$$

Let $\hat{x}_0(k)$ denote the optimal solution obtained by the approximate algorithm.

First we need the following two lemmas.

Lemma 5.1: Let $L_n(\theta) = \ln f(Z_n;\theta)$. Suppose the system (1.1)-(1.2) is asymptotically stable and Kalman filter matched to any $\theta \in \Omega = \Omega_1 \cup \Omega_2$ is asymptotically stable. Then

$$\lim_{n \to \infty} \frac{1}{n} L_n(\theta) = L_\infty(\theta) \quad \text{a.e. } P_{\theta_o}^{Z^\dagger} \text{ uniformly in } \theta \in \Omega,$$

where

$$L_\infty(\theta) = \lim_{n \to \infty} E\{\ln f(z_n|Z_{n-1};\theta)|\theta_o\}$$

and where the expectation is taken over the random variables z_j, $j = 1,2,\ldots$, assuming that they obey (1.1)-(1.2) with $\theta = \theta_o$, the true parameter value.

Proof: See Kashyap [32]. (See also [63] and [15].)

Lemma 5.2: Suppose that conditions of Lemma 5.1 are satisfied. Let $S_p(\theta) \triangleq \frac{\partial}{\partial \theta_p} L_\infty(\theta)$, $p = 1,2,\ldots,q$, be continuous for $\theta \in \Omega = \Omega_1 \cup \Omega_2$, where θ_p denotes the p^{th} component of θ. Assume that $S_p(\theta) = 0$, $p = 1,2,\ldots,q$, has a unique solution $\theta = \theta_0$ and that θ_0 maximizes $L_\infty(\theta)$, where θ_0 is the true parameter value. Then $L_\infty(\theta)$ is a monotone increasing function of θ_p for $\theta_p \leq \theta_{0p}$ and a monotone decreasing function of θ_p for $\theta_p \geq \theta_{0p}$.

---

† $P_{\theta_o}^{Z}$ is a probability measure induced on the range space of the random sequence $\{z_0,z_1,z_2,\ldots\}$ indexed by the true parameter value θ_o.

Proof: It follows trivially from the assumptions.

Uniqueness of the root of $S_p(\theta)$ at $\theta = \theta_0$ is a consequence of the uniqueness and consistency of the maximum likelihood estimate of θ [15,32].

Now we present the convergence results.

Theorem 5.1. Let Ω_1 and Ω_2 be disjoint bounding sets where $\Omega_1 = \{\theta_{11} \le \theta \le \theta_{12}\}$, $\Omega_2 = \{\theta_{21} \le \theta \le \theta_{22}\}$, $\theta_{12} < \theta_{21}$ and where the vector inequality holds componentwise. Suppose $E\{\|x_n - \hat{x}(n)\|^2 | \theta\}$ is a monotone increasing function of θ for all n. Suppose the conditions of Lemmas 5.1 and 5.2 are satisfied. Let $\theta_0 \in \Omega_1$ where θ_0 is the true parameter value. Then the following holds:

(i) If $f(Z_n;\theta_{21}) < f(Z_n;\theta_{11})$ for $n \ge$ some n_0, then $\lim_{n \to \infty} \hat{x}_0(n)$ is a MMSE estimate matched to θ_{12}.

(ii) If $f(Z_n;\theta_{21}) \ge f(Z_n;\theta_{11})$ for $n \ge$ some n_0, then in $\lim_{n \to \infty} \hat{x}_0(n)$ either $\theta_2 = \theta_{21}$, $\theta_1 = \theta_{12}$, or $\theta_2 = \theta_2'$, $\theta_1 = \theta_1'$ (see Equation (5.15)) where θ_i', i = 1,2, are such that $L_\infty(\theta_1') = L_\infty(\theta_2')$. In the former case $\lim_{n \to \infty} \hat{x}_0(n)$ is a MMSE estimate matched to θ_{12}.

Proof: From Lemma 5.1, given $\delta > 0$ $\exists$ $N(\delta)$ such that

$$\left|\frac{1}{n} L_n(\theta) - L_\infty(\theta)\right| \le \delta/2 \qquad \text{a.e. } P^Z_{\theta_o}$$

for $n \ge N(\delta)$ uniformly in $\theta \in \Omega$. Therefore, we have

$$-\frac{\delta}{2} + L_\infty(\theta_i) \le \frac{1}{n} L_n(\theta_i) \le \frac{\delta}{2} + L_\infty(\theta_i), \quad i = 1,2,; \quad \theta_i \in \Omega_i .$$

Hence it follows that

$$L_\infty(\theta_1) - L_\infty(\theta_2) - \delta \le \frac{1}{n}\left[L_n(\theta_1) - L_n(\theta_2)\right]$$

$$\le L_\infty(\theta_1) - L_\infty(\theta_2) + \delta$$

which in turn results in

$$\exp[n\{L_\infty(\theta_1) - L_\infty(\theta_2) - \delta\}] \le f(Z_n;\theta_1)/f(Z_n;\theta_2)$$

$$\le \exp[n\{L_\infty(\theta_1) - L_\infty(\theta_2) + \delta\}] \ .$$

Now, since $L_\infty(\theta)$ is a real number, one of the following relations must be true:

$$L_\infty(\theta_1) > L_\infty(\theta_2),\ L_\infty(\theta_1) = L_\infty(\theta_2) \text{ or } L_\infty(\theta_1) < L_\infty(\theta_2).$$

Suppose that $L_\infty(\theta_1) > L_\infty(\theta_2)$. Then $\exists\ \epsilon > 0 \ni\ L_\infty(\theta_1) = L_\infty(\theta_2) + \epsilon$.

Choose $\delta < \epsilon$. Then

$$\frac{f(Z_n;\theta_1)}{f(Z_n;\theta_2)} \ge \exp[n(\epsilon - \delta)] \text{ for } n \ge N(\delta)$$
$$\to \infty \text{ as } n \to \infty.$$

Similarly for $L_\infty(\theta_1) < L_\infty(\theta_2)$, we have $\lim_{n \to \infty} [f(Z_n;\theta_1)/f(Z_n;\theta_2)] = 0$.

(For $L_\infty(\theta_1) = L_\infty(\theta_2)$ we cannot follow this approach.)

(i) Since $f(Z_n;\theta_{21}) < f(Z_n;\theta_{11})$ for $n \ge n_o$,

$$\lim_{n \to \infty} [f(Z_n;\theta_1)/f(Z_n;\theta_2)] \to \infty \quad \forall\ \theta_1,\ \theta_2.$$

Since $E\{\|x_n - \hat{x}(n)\|^2 | \theta\}$ is a monotone increasing function of θ, $\lim_{n \to \infty} L_1$ (see Equation (5.11)) is maximized for $\theta_1 = \theta_{12}$. Hence $\lim_{n \to \infty} \hat{x}_0(n)$ is a MMSE estimate matched to θ_{12}.

(ii) Now $f(Z_n;\theta_{21}) \geq f(Z_n;\theta_{11})$ for $n \geq$ some n_0. For $\theta_0 \in \Omega_1$, in $\lim_n \hat{x}_0(n)$, θ_1, θ_2 such that $L_\infty(\theta_1) < L_\infty(\theta_2)$ is not possible since we can always choose some θ_1 "closer" to θ_0 in order to maximize L_1 in (5.11) for a fixed θ_2. Therefore, in $\lim_{n \to \infty} \hat{x}_0(n)$, θ_1 and θ_2 are such that either $L_\infty(\theta_1) > L_\infty(\theta_2)$ or $L_\infty(\theta_1) = L_\infty(\theta_2)$. In the former case, as in part (i) of this theorem, $\lim_{n \to \infty} \hat{x}_0(n)$ is a MMSE estimate matched to θ_{12}.

Q.E.D.

Theorem 5.2: Let the hypotheses of Theorem 5.1 hold except that now $\theta_0 \in \Omega_2$. Then the following holds:

(i) If $f(Z_n;\theta_{22}) > f(Z_n;\theta_{12})$ for $n \geq$ some n_0, then $\lim_{n \to \infty} \hat{x}_0(n)$ is a MMSE estimate matched to θ_{22},

(ii) If $f(Z_n;\theta_{22}) \leq f(Z_n;\theta_{12})$ for $n \geq$ some n_0, then in $\lim_{n \to \infty} \hat{x}_0(n)$, θ_1 and θ_2 are such that $L_\infty(\theta_1) \leq L_\infty(\theta_2)$. Further, these θ_1 and θ_2 are such that $\theta_0 - \theta_1 > \underline{0}$ and $\theta_2 - \theta_0 > \underline{0}$ where $\underline{0}$ is a q-vector with all zero entries.

Proof: The proof is similar to that given for Theorem 5.1; therefore, it is omitted.

An interpretation of these theorems is that the optimal estimate converges to one of the following two estimates: a) An estimate matched (optimal in the MMSE) to a value of the parameter in the same region as the true value independent of λ_i's, i = 1,2. b) A weighted average of two estimates one in each region, with the performance (CMSE) of the one in the wrong region at least as good as the performance of the one in the correct region.

An example of the case where $E\{\|x_n - \hat{x}(n)\|^2|\theta\}$ is a monotone increasing function of θ for all n is the problem where uncertainty lies only in the diagonal elements of matrices Q and/or R.

5.5. Example

Consider the following scalar dynamic system:

$$x_{k+1} = 0.9\, x_k + w_k$$

$$z_k = x_k + v_k, \quad k = 0,1,2,\ldots$$

with $E\{x_0\} = 1.0$, $E\{x_0^2\} = 3.0$, $E\{v_k^2\} = R = 0.2$, $E\{w_k^2\} = Q$. The unknown parameter is assumed to be Q and it belongs to either Ω_1 or Ω_2 where $\Omega_1 = [0.05,0.2]$ and $\Omega_2 = [0.2,1.0]$. The bounding sets are quantized as follows: $\Omega_1 = \{0.1,0.2\}$ and $\Omega_2 = \{0.2,0.4,0.6,0.8,1.0\}$. Thus 6 Kalman filters are run in parallel. A computer simulation was carried out using random number generators. In Figures 5.1 and 5.3 plots for r.m.s. errors versus true parameter values are given for stages k = 3,7,15 respectively, averaged over 100 Monte Carlo runs. In each of these figures, plots for r.m.s. errors due to (i) the proposed scheme (ii) the minimax scheme (iii) the adaptive scheme due to Magill [36] and (iv) optimal Kalman filtering assuming knowledge of the true parameter values, are given. In the proposed scheme we choose $\lambda_1 = \lambda_2 = 0.5$. In the adaptive scheme it is assumed that Q is equiprobable within each Ω_i, i = 1,2 and that the probability $Q \in \Omega_i$ equals λ_i, i = 1,2.

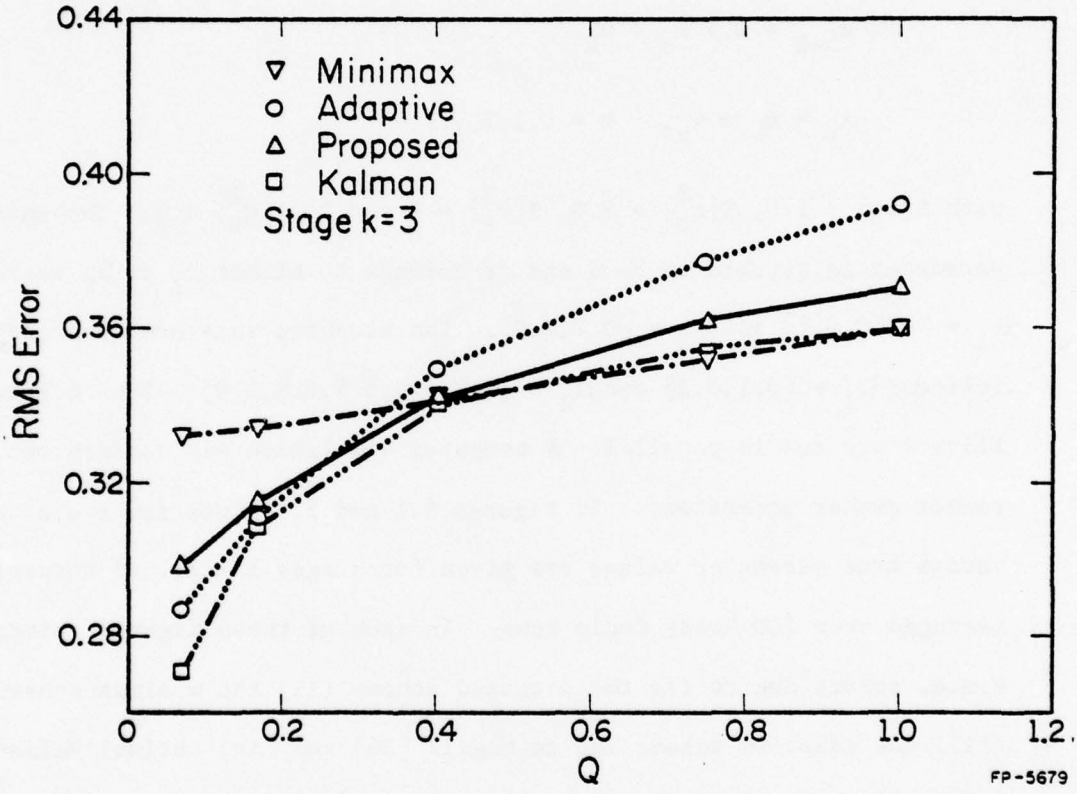

Figure 5.1. RMS error in state estimate vs. unknown parameter for k=3.

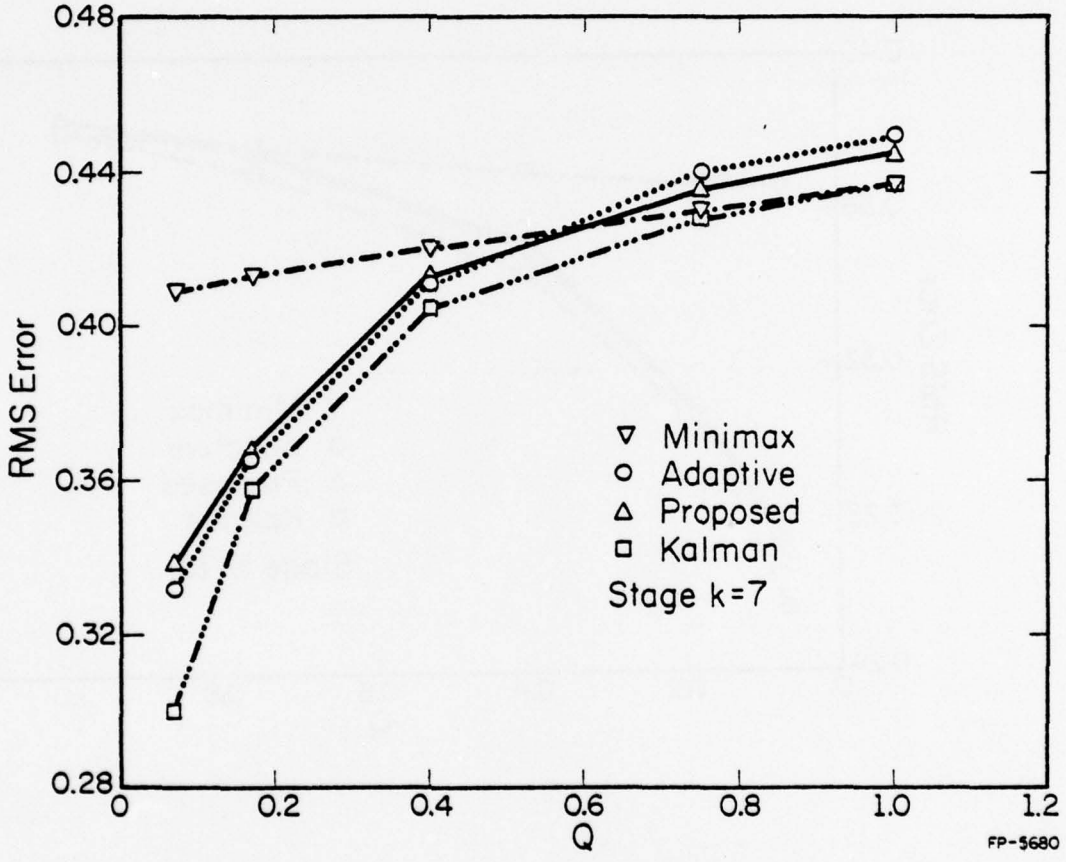

Figure 5.2. RMS error in state estimate vs. unknown parameter for k=7.

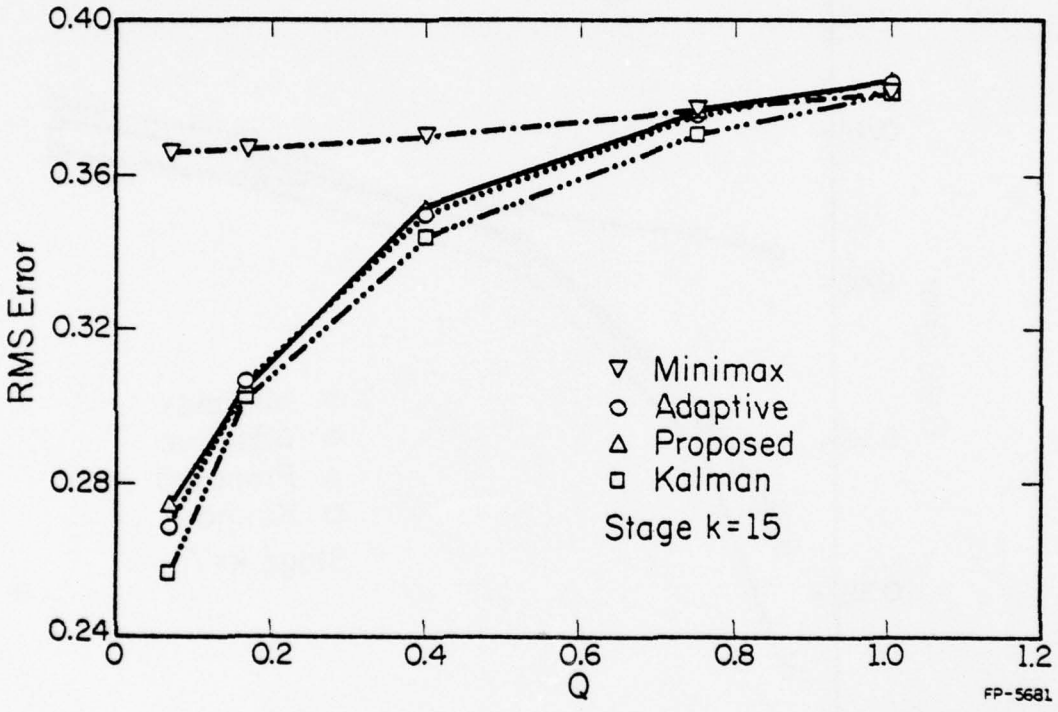

Figure 5.3. RMS error in state estimate vs. unknown parameter for k=15.

From these figures, we note the following. In comparison with the minimax scheme, the proposed scheme decreases the maximum MSE (mean-squared error) in region Ω_1 at the cost of some increment in the maximum MSE in Ω_2. This increment in Ω_2, however, decreases with time whereas improvement in Ω_1 gets better. Further, comparison with the adaptive (Bayesian) scheme shows that the small-sample performance (see Fig. 5.1) of the proposed scheme is superior. Also, adaptive features of the proposed scheme become obvious from Figs. 5.2 and 5.3. Thus this example tends to support the objective of the proposed approach--to reduce the pessimism of the standard minimax scheme for large observation record without affecting the small-sample performance.

5.6. Discussion

A method of state estimation in uncertain linear dynamical systems with noise covariances unknown has been presented. It is a heuristic extension of the standard minimax scheme to the case when multiple bounds on the unknown parameters are available. The estimator is shown to converge under certain conditions to a MMSE estimator matched to a value of the unknown parameter which lies in the same bounding set as does the true parameter value. An example of a scalar dynamical system is given which tends to support the usefulness of the proposed approach in reducing the pessimism of the standard minimax scheme for large observation record and large uncertainties, while retaining the desirable small-sample performance.

The proposed scheme differs from Padilla's approach [45,46] in that we consider a weighted conditional mean-square error cost function given the past observations rather than simply a weighted mean-square error cost function. Extension of Padilla's approach to dynamical systems seems to be extremely cumbersome computationally. Furthermore, it is not at all clear as to how Padilla's approach could be extended to the time-varying parameter case. In the next chapter we consider extension of the proposed scheme to the case of time-varying uncertainties.

CHAPTER 6

ESTIMATION FOR SYSTEMS WITH UNKNOWN, TIME-VARYING NOISE STATISTICS

6.1. Introduction

In the last chapter we investigated a detection-estimation scheme for state estimation in linear dynamical systems with unknown, time-invariant noise covariances. In this chapter the scheme is extended to systems with unknown, time-varying noise covariances. Past work on this type of problem has been quite ad-hoc. It consists mainly of heuristic extensions of the schemes optimal for the time-invariant uncertainty case. For example, in Bar-Shalom et.al. [7], a fading memory filter is proposed to track the time variations in the unknown noise covariances. A similar approach may also be found in Alspach [3]. The "degree" of fading is taken to be a design parameter, and the scheme requires on-line "tuning". Such schemes appear to be able to track slow variations in unknown parameters quite well, but may run into serious difficulties if the unknown parameters vary rapidly, precluding enough time for the filter to adjust to the changes. In some other approaches, such as [11] and [9], it is assumed that a state space model, driven by white noise, of the nonstationary covariance parameters is available. In most of the applications this may turn out to be quite restrictive.

The proposed approach too is based on heuristic considerations; however, considerable attention is paid to small-sample performance also, so as to enable the estimator to handle both fast as well as slow variations in the unknown parameters. The proposed scheme is discussed in Section 6.2. In Section 6.3, an example is considered to illustrate the performance of the scheme.

6.2. Proposed Scheme

The system model under consideration is the same as in Section 5.3, namely, Case 3 of model (1.1)-(1.2), except that now the uncertainties are allowed to be time-varying. Assume that the given model is specified up to a set of unknown parameters, possibly time-varying, denoted by the q-vector $\theta(k)$ at time k. As in Chapter 5 this uncertainty may be in noise covariances Q_k or R_k (at time instant k) only, which are assumed to be continuous functions of $\theta(k)$. The unknown parameter vector $\theta(k)$ is assumed to satisfy at least one of the following conditions:

$$\theta(k) \in \Omega_i, \qquad i = 1,2,\ldots,N$$

where Ω_i, $i = 1,2,\ldots,N$ is a collection of compact subsets of $\mathbf{R}^q$. An example of such conditions is given below:

$$\Omega_i^{k+1} = \{\Theta(k): \underline{a} \le \theta(k) \le \underline{b},^{\dagger}\ |\theta(k+1)-\theta(k)| \le \underline{d}_i,^{\dagger}\ \forall\ k\}$$

$$i = 1,2;\quad N = 2;\quad \Omega_i^{k+1} \triangleq \Omega_i x \ldots x\, \Omega_i,\ k+1 \text{ times}$$

where $\underline{a}$, $\underline{b}$, $\underline{d}_i$ are time-invariant q-vectors; $\underline{d}_1 < \underline{d}_2$ and $\underline{0}$ denotes a q-vector with all zero entries; and $\Theta(k) \triangleq \{\theta(j),\ 0 \le j \le k\}$ denotes a

---

†The inequality is satisfied componentwise.

parameter sequence. The overall region in which $\theta(k)$ lies is known. Within that set it varies either slowly ($|\theta(k+1)-\theta(k)| \leq \underline{d}_1$) or can also change rapidly ($|\theta(k+1)-\theta(k)| \leq \underline{d}_2$). Another example is,

$$\Omega_i^{k+1} = \{\Theta(k): \ \underline{a} \leq \theta(k) \leq \underline{b}_i, \ \forall\ k\}$$
$$i = 1,2; \quad N = 2; \quad \underline{b}_1 < \underline{b}_2$$

In this case there is no restriction on the rate of change of $\theta(k)$. Depending upon the problem at hand, several variations of the Ω_i's are possible. Also, notice that in these examples we have used nested sets, that is, $\Omega_1 \subset \Omega_2$.

The objective is to obtain an estimate of the value of x_k based on observations $Z_k = \{z_j, 0 \leq j \leq k\}$.

We follow the approach already discussed in Chapter 5 except that, at stage k, instead of time-invariant θ, we use parameter sequence $\Theta(k)$. As in Chapter 5, let $\hat{x}(k)$ denote an estimate of x_k based on the observation Z_k. The CMSE of this estimate given the observation sequence Z_k and the parameter sequence $\Theta(k)$ is given by

$$L[\hat{x}(k),Z_k,\Theta(k)] = E\{\|x_k - \hat{x}(k)\|^2 | Z_k,\Theta(k)\} \tag{6.1}$$

The optimal estimator is defined as the one that minimizes a weighted sum of maximum errors given as

$$\sum_{i=1}^{N} \max_{\Theta(k) \in \Omega_i^{k+1}} \{\lambda_i' \ L[\hat{x}(k),Z_k,\Theta_i(k)]\} \tag{6.2}$$

where

$$\lambda_i' = \lambda_i f(Z_k|\Theta_i(k))/[\sum_{j=1}^{N} \lambda_j f(Z_k|\Theta_j(k))] \tag{6.3}$$

$$\sum_{i=1}^{N} \lambda_i = 1, \quad \lambda_i > 0, \quad i = 1,2,\ldots,N$$

$f(Z_k|\Theta_i(k))$ = conditional density function of Z_k given the parameter sequence $\Theta_i(k)$

and $\Theta_i(k)$ implies that $\Theta(k) \in \Omega_i^{k+1}$. The remarks made in Section 5.2 apply here too. Now the problem is as given in (6.4),

$$\min_{\hat{x}(k) \in S} \sum_{i=1}^{N} \max_{\Theta_i(k) \in \Omega_i^{k+1}} [\lambda_i' L[\hat{x}(k), Z_k, \Theta_i(k)]] \tag{6.4}$$

where S denotes the space of all measurable transformations from the observation space Z_k into $\mathbb{R}^n$. Since an exact solution to problem (6.4) appears to be intractable, an ad-hoc solution, given by (6.5), is investigated as in Section 5.2.

$$\hat{x}(k) = \sum_{i=1}^{N} \lambda_i' \hat{x}_i(k) \tag{6.5}$$

where $\hat{x}_i(k)$ denotes the MMSE estimator matched to the sequence $\Theta_i(k)$ and based on data Z_k. The efficacy of such a solution for the constant parameter case was demonstrated in Chapter 5.

The problem now becomes that of finding $\Theta_i(k)$ to maximize

$$L_i \triangleq \lambda_i' E\{\|x_k - \hat{x}(k)\|^2 | Z_k, \Theta_i(k)\} \tag{6.6}$$

$$= \lambda_i' E\{\|x_k - \hat{x}_i(k)\|^2 | Z_k, \Theta_i(k)\} + \lambda_i' \|\hat{x}(k) - \hat{x}_i(k)\|^2 \tag{6.7}$$

$$= \lambda_i' E\{\|x_k - \hat{x}_i(k)\|^2 | \Theta_i(k)\} + \lambda_i' \|\hat{x}(k) - \hat{x}_i(k)\|^2 \tag{6.8}$$

When $\Theta_i(k)$, $i = 1,2,\ldots,N$ are known, L_i can be generated sequentially in time using N Kalman filters one each matched to $\Theta_i(k)$.

Approximate Solution

In general no closed-form analytical solution for maximizing $\Theta_i(k)$'s exists. So we have to resort to a numerical solution. First the continuous parameter space is approximated with a finite set of quantized points. Let $\theta(k) \in \Omega_i$ be quantized into m_i discrete values $\theta_j(k)$, $j = 1,2,\ldots,m_i$; $i = 1,2,\ldots,N$. Then, at stage k, we have (at most) m_i^{k+1} possible sequences $\Theta_i(k)$ in Ω_i^{k+1}, $i = 1,2,\ldots,N$; $k = 0,1,2,\ldots$, ignoring other possible constraints such as bounds on rate of change of $\theta(k)$, etc. Let $\Theta_{i\ell}(k)$ denote a specific sequence in Ω_i^{k+1} at stage k. Then we have a number of Kalman filters one each matched to $\Theta_{i\ell}(k)$, $\ell = 1,2,\ldots,m_i^{k+1}$; $i = 1,2,\ldots,N$. This gives us $\hat{x}_{i\ell}(k)$ for each possible i and ℓ, where the subscript $i\ell$ on $\hat{x}(k)$ denotes an MMSE estimator matched to the sequence $\Theta_{i\ell}(k)$. As a byproduct we obtain $E\{\|x_k - \hat{x}_{i\ell}(k)\|^2 | Z_k, \Theta_{i\ell}(k)\}$ and the probability $f(Z_k|\Theta_{i\ell}(k))$ for all i and ℓ. All these computations can be performed recursively (see, for example, [57]).

Now L_i can be found for all possible $\Theta_i(k)$'s. For every k, find

$$\Theta_{i0}(k) = \arg\{ \max_{\Theta_i(k) \in \Omega_i^{k+1}} L_i \}, \quad i = 1,2,\ldots,N \tag{6.9}$$

Note that here we are solving N coupled expressions. Substituting these $\Theta_{i0}(k)$, $i = 1,2,\ldots,N$ in (6.5) gives the required estimator $\hat{x}(k)$.

Reduction in Computational Complexity

The computation and storage requirements of the proposed approximate solution increase exponentially with time. We shall now discuss some further approximations which will be used to control the number of Kalman filters to be a finite, or below a maximum allowable, number. The procedure resembles somewhat the detection-estimation approach discussed in Chapter 2. Instead of considering all the probability densities $f(Z_k|\Theta_{ij}(k))$, $i = 1,2,\ldots,N$, at stage k, corresponding to all possible parameter sequences $\Theta_{ij}(k)$, we shall disregard some of the "unlikely" sequences, i.e., not process the Kalman filters matched to these unlikely sequences. Also, if the densities corresponding to two distinct sequences are "close" to each other in certain distance measure, only one of them need be considered.

To develop an editing criterion to limit the number of "modal" filters, we need a distance measure. A measure of distance between two probability densities $f_i(x)$ and $f_j(x)$ is the Bhattacharyya coefficient (B-coeff.) ρ_{ij} given by

$$\rho_{ij} = \int_{-\infty}^{\infty} \sqrt{f_i(x) f_j(x)}\, dx,$$

and the B-distance is given by $-\ell n \rho_{ij}$. Note that $0 \leq \rho_{ij} \leq 1$ and for $f_i(x) = f_j(x)$, $\rho_{ij} = 1$. The reasons for choosing B-coefficient as distance measure are that (i) it is closely related to the performance bounds in detection and estimation theory [29] and (ii) it is easy to compute for Gaussian densities.

In order to establish criteria according to which we make decisions to drop a parameter sequence we need a threshold α where $0 < \alpha < 1$. Now two parameter sequences $\Theta_{ij}(k)$ and $\Theta_{i\ell}(k)$ will be considered <u>distinguishable</u> iff the B-coefficient between the densities $f(Z_k|\Theta_{ij}(k))$ is <u>less</u> than α. Then we have the following algorithm to find the "best" state estimate. It is given, for simplicity, only for the case of two (nested) bounding sets Ω_i, $i = 1,2$; $\Omega_1 \subset \Omega_2$. Extension to larger number of sets is straightforward.

<u>Algorithm</u>

Let

M = Maximum number of Kalman filters to be processed at any time k.

M_1 = Maximum number of Kalman filters matched to parameter sequences in Ω_1^{k+1} at any time k.

M_2 = Maximum number of Kalman filters matched to parameter sequences in Ω_2^{k+1} at any time k.

= M since $\Omega_1 \subset \Omega_2$

<u>Step 1</u>. Compute the CMSE L_{ij}, $i = 1,2$, for all admissible parameter sequences $\Theta_{ij}(k)$ at time k where L_{ij} denotes the CMSE L_i, given by Equation (6.8), computed for a specific sequence $\Theta_{ij}(k)$. Find the pair of sequences $(\Theta_{10}(k),\Theta_{20}(k))$, $\Theta_{i0}(k) \in \Omega_i^{k+1}$, $i = 1,2$, which maximizes L_i, $i = 1,2$, given by Equation (6.9).

Step 2. Now select the sequence $\Theta_{ij}(k) \in \Omega_i^{k+1}$, $\Theta_{ij}(k) \neq \Theta_{i0}(k)$, which gives the largest L_{ij} closest to L_{i0} where L_{i0} is computed using sequence $\Theta_{i0}(k)$. Check if $\rho_{j0}^{i} < \alpha$ where ρ_{j0}^{i} is the B-coefficient between densities $f(Z_k|\Theta_{ij}(k))$ and $f(Z_k|\Theta_{i0}(k))$, $i = 1,2$. If true, then this parameter sequence is selected as the sequence corresponding to the "second" Kalman filter of the maximum allowed number M_i for set Ω_i^{k+1}. If false, then select the $\Theta_i(k)$ which gives the next largest CMSE and repeat the procedure. The process is continued until either the number of selected sequences equal M_i, $i = 1,2$, for regions Ω_i^{k+1}, $i = 1,2$, respectively, or all admissible sequences have been considered, whichever occurs earlier. Let M_i^i denote the number of sequences selected in region Ω_i^{k+1}, $i = 1,2$.

Step 3. Set $k = k+1$ and go to Step 2, with the exception that now $\Theta_{ij}(k+1)$ are all possible "extensions" of $\Theta_{i\ell}(k)$, where $\ell = 1,2,\ldots,M_i^i$; $M_i^i \leq M_i$, $i = 1,2$, consistent with the specifications of the set Ω_i^{k+2}, $i = 1,2$.

Remark 6.1. The integers M_i, $i = 1,2$, are design parameters, usually dictated by the computation and storage capabilities of the processor, and accuracy desired. Also, the threshold α is selected empirically.

Remark 6.2. In Step 2, all distances are measured from the selected "optimum" pair $(\Theta_{10}(k),\Theta_{20}(k))$.

Remark 6.3. The scheme given above includes the one-step-at-a-time approach in which only two Kalman filters corresponding to $\Theta_{i0}(k)$, $i = 1,2$, are carried forward to stage $k+1$.

In the next section we consider an illustrative numerical example.

6.3. Example

To illustrate the feasibility of the scheme, a simple system was simulated. The dynamics of the system is represented by the scalar state equation

$$x_{k+1} = 0.98\, x_k + w_k$$

and the observation equation is

$$z_k = x_k + v_k, \quad k = 0,1,2,\ldots \quad .$$

The initial state is $x_0 \sim N(10,100)$, whereas for simulation the observation samples were obtained using $x_0 = 1$. Furthermore, $v_k \sim N(0,0.5)$ and $w_k \sim N(0,Q_k)$ $\forall$ k. The driving noise covariance Q_k is assumed to be unknown. It is further assumed that Q_k is time-varying and is known to belong to either Ω_1 or Ω_2, where $\Omega_1 = \{Q_k : Q_k = 1,2,4\}$ and $\Omega_2 = \{Q_k : Q_k = 1,2,4,6,8,10\}$. The bounding sets Ω_i, i = 1,2, may be thought of as resulting from quantization of the continuous sets $\{Q_k : Q_k \leq 4.0\}$ and $\{Q_k : Q_k \leq 10.0\}$, respectively. There is no restriction on the rate of change of Q_k.

The system was simulated using random number generators, and filtering was carried through 59 stages. The "true" driving noise covariance Q_k was chosen as given below:

$$Q_k = \begin{cases} 1.0 & \text{for } 0 \leq k \leq 9 \\ 3.0 & \text{for } 10 \leq k \leq 29 \\ 8.0 & \text{for } 30 \leq k \leq 44 \\ 1.0 & \text{for } 45 \leq k \leq 59 \end{cases}$$

Figure 6.1 shows the true covariance Q_k vs. time. The performances of various schemes (described below) were computed by averaging over 100 Monte Carlo runs. The same sequence of true noise covariance Q_k was

used for all the runs. The proposed detection-estimation approach was implemented using $\lambda_1 = \lambda_2 = 0.5$, $M_1 = 5$ and $M_2 = 10$. Therefore, we carry forward at most 5 Kalman filters in Ω_1^k and 10 Kalman filters in Ω_2^k. Furthermore, at any time k, we consider at most 15 extensions of M_1 sequences in Ω_1^{k+1} and 60 extensions of M_2 sequences in Ω_2^{k+1}. The B-distance feature was not incorporated. In Fig. 6.2, plots for "instantaneous" r.m.s. error in state estimate vs. time are given for a typical observation sample sequence for the two approaches: proposed detection-estimation scheme, and optimal Kalman filter using the true covariance sequence $\{Q_k\}$.

In Fig. 6.3, plots for average r.m.s. error in state estimate vs. time are shown, averaged over 100 runs, for the following schemes: (i) *proposed detection-estimation scheme*, (ii) optimal Kalman filter using true covariance sequence $\{Q_k\}$, (iii) fixed parameter Kalman filter with fixed covariance value given by $Q_k = 8.0 \; \forall \; k$. It is clear from Fig. 6.3 that the proposed scheme is capable of "tracking" the system state very well for relatively high values of Q_k; see the performance over $10 \leq k \leq 44$. For low values of Q_k (i.e., $Q_k = 1.0$), the state estimate due to the detection-estimation scheme has relatively high r.m.s. error compared to that due to optimal Kalman filter with true covariance sequence. However, the detection-estimation scheme is much better than the fixed parameter Kalman filter with covariance $Q_k = 8.0 \; \forall \; k$.

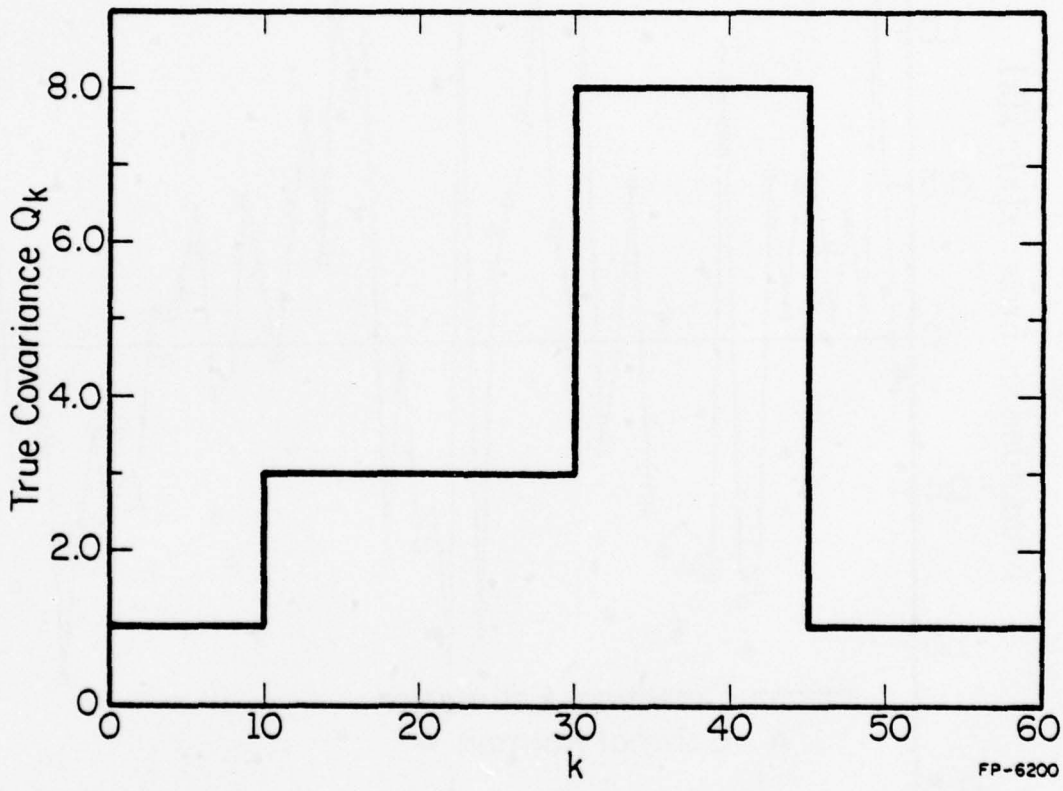

Figure 6.1. True covariance Q_k vs. time k.

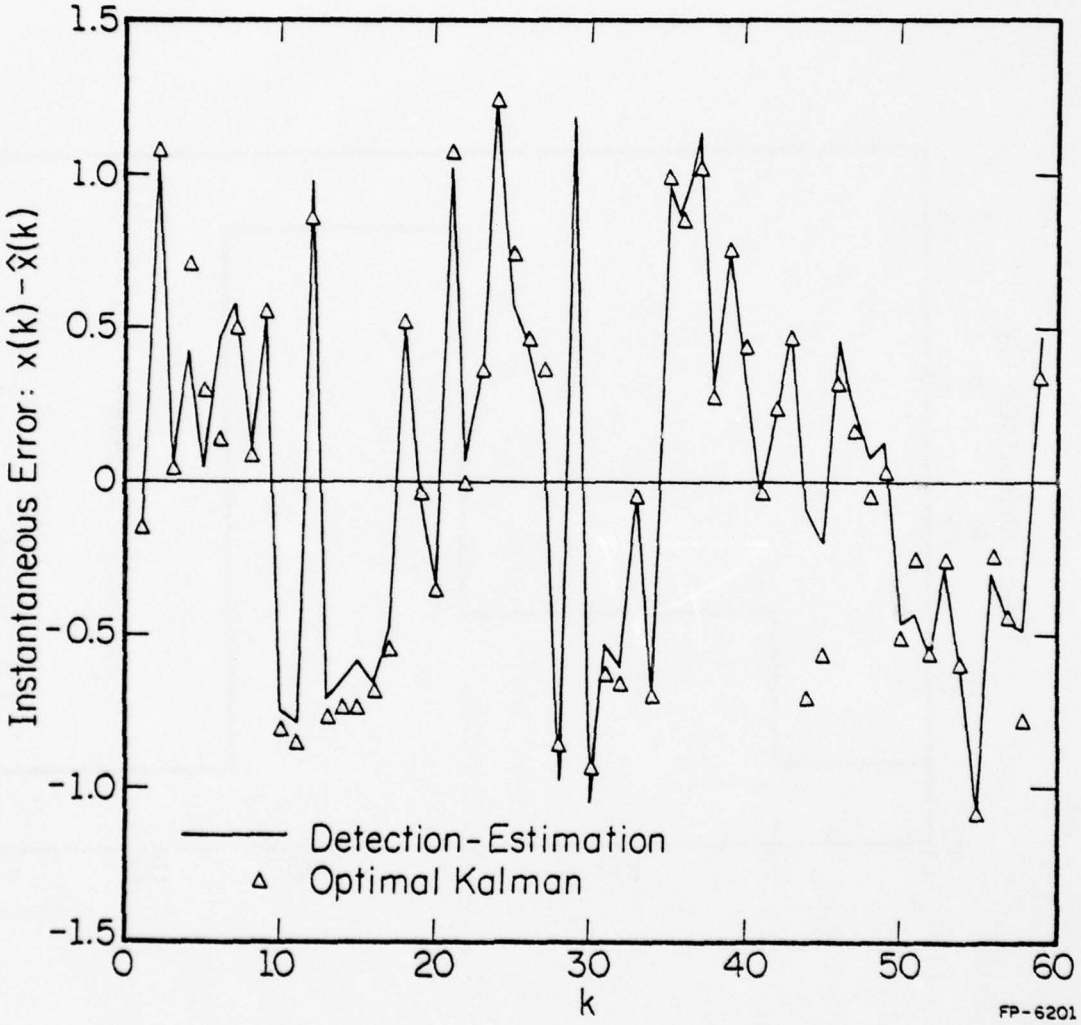

Figure 6.2. Instantaneous state estimation error vs. time for a typical observation sample sequence.

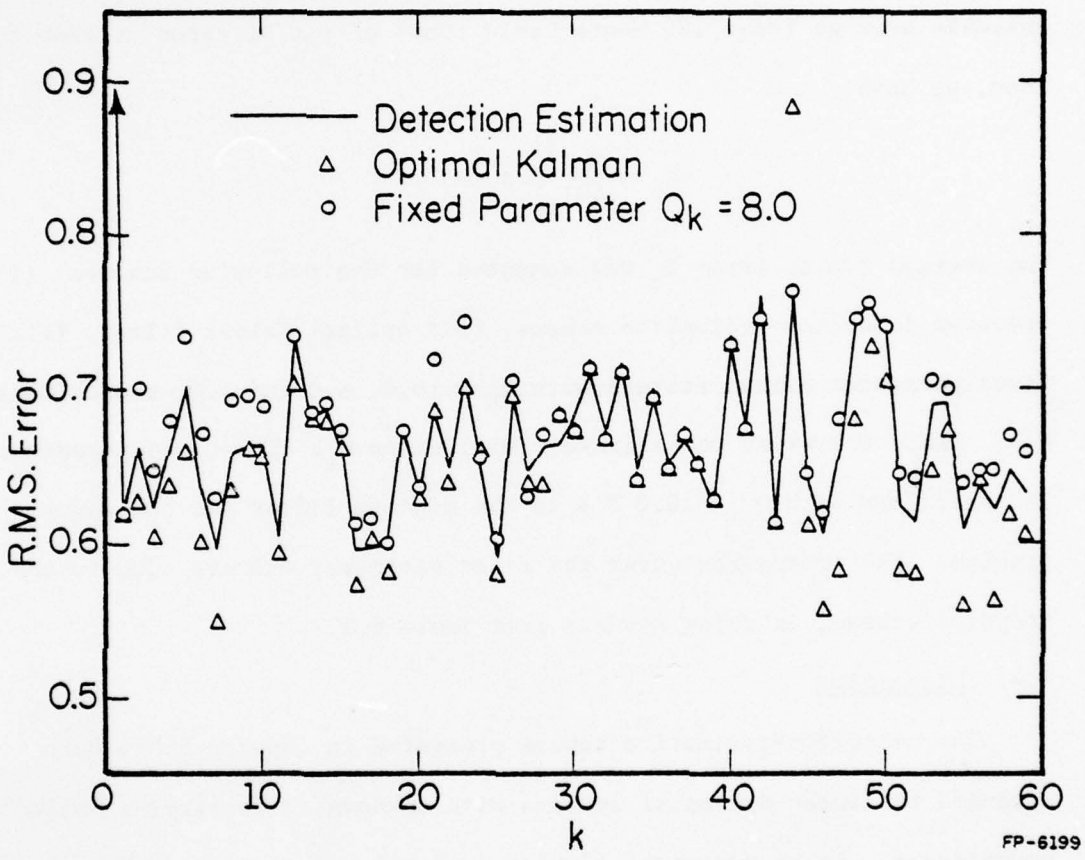

Figure 6.3. Comparison of the RMS state estimation errors.

In Table 6.1, the combined ensemble- and time-averages of r.m.s. errors due to various schemes are given. The r.m.s. error in the state estimate was first averaged over 100 Monte Carlo runs, then it was averaged over 59 stages. Let D_t denote the combined ensemble- and time-average of r.m.s. error in the state estimate, and let $D_e(k)$ denote the ensemble average (over 100 Monte Carlo runs) of r.m.s. error at time k. Then, we have

$$D_t = \frac{1}{59} \sum_{k=1}^{59} D_e(k) \ .$$

The average r.m.s. error D_t was computed for the following schemes: (i) proposed detection-estimation scheme, (ii) optimal Kalman filter, (iii) fixed parameter Kalman filters with Q_k = 10.0, 8.0 and 1.0, respectively, $\forall$ k. Table 6.1 gives the average r.m.s. error D_t. The fixed parameter Kalman filter with Q_k = 10.0 $\forall$ k is the minimax filter for the given problem. The improvement over the fixed parameter filters, due to the proposed scheme, is quite obvious from Table 6.1.

6.4. Discussion

The detection-estimation scheme presented in Chapter 5 has been extended to linear dynamical systems with unknown, time-varying noise covariances. An approximate solution is developed to reduce the computational complexity of the proposed scheme. A numerical example of a scalar dynamical system is considered to illustrate the feasibility of the scheme. The motivations for considering the proposed approach are the same as in Chapter 5, namely, reduction in the pessimism of the standard minimax

Table 6.1

Comparison of the average r.m.s. error D_t due to various schemes.

Scheme	Average r.m.s. error D_t
Optimal Kalman	0.6229
Detection-Estimation	0.6621
Fixed Parameter Kalman Filter with	
$Q_k = 10.0$	0.6765
$Q_k = 8.0$	0.6728
$Q_k = 1.0$	0.6933

scheme for large uncertainties and large observation record while retaining its desirable small-sample performance. A good small-sample performance should enable the proposed scheme to "track" well rapid variations in the noise covariances, whereas the schemes relying mainly on good asymptotic performance do not have enough time to adjust to the rapid changes.

We have not compared the proposed scheme to any other approach to state estimation for systems with unknown, time-varying noise covariances, such as given in [3] and [7]. The main reason for this is that these approaches need parameters which can only be found empirically by "tuning" the filters on-line. Therefore, any implementation of these schemes needs extensive simulations to select suitable values for such parameters. Moreover, the values of these parameters depend upon the rate of change of unknown, time-varying uncertainties. Such a comparison is, nevertheless, of considerable interest and is left for future research. Finally, we note that more extensive simulations are needed to judge the suitability of the proposed detection-estimation approach to state estimation for systems with time-varying covariances.

CHAPTER 7

SUMMARY AND CONCLUSIONS

In this thesis the problem of state estimation for a class of linear discrete-time dynamical systems with unknown time-varying parameters has been studied. The basic objective was to model the time-variations of the unknown parameters in an appropriate fashion to render the problem analytically and computationally tractable, while, at the same time, to employ models general enough to describe a large class of time-varying characteristics of the system parameters. Another objective was to develop estimation algorithms for the chosen models and to investigate their properties, specifically, to examine their asymptotic behavior. Attention was focused mainly on systems with unknown time-varying noise statistics. Two different approaches to modeling and estimation under time-varying uncertainties were investigated. In one of the approaches, the jump parameters were assigned a probabilistic description. In the other approach, multiple bounds on the unknown parameter values and its time derivatives were assumed to be available.

In Chapters 2 through 4 we used a finite state Markov chain model for the jump parameters. The parameters were assumed to take values only from a finite set, with transitions from one value to another determined by a Markov transition probability matrix. In Chapter 2 the transition matrix was assumed to be known. A Bayesian approach was adopted for MMSE state estimation and the optimal estimator was found to require exponentially increasing computation and storage capabilities with time. A new

suboptimal approach, employing a detection-estimation scheme, was proposed in Chapter 2 to alleviate the excessive computational requirements of the optimal state estimator. Two simulation examples were presented which indicated the superiority of the proposed detection-estimation approach over some of the existing suboptimal algorithms. The approaches discussed in Chapter 2 are applicable to general dynamical models with jump parameters. However, in Chapter 3, we confined our attention to switchings in the noise statistics only.

In Chapter 3, an adaptive Bayesian MMSE estimation scheme was investigated, both analytically and through computer simulation. The Markov transition probability matrix was assumed to be unknown and to belong to a finite set which also contained the true transition matrix. The objectives in this chapter were to present feasible adaptive algorithms that sequentially produced an approximate MMSE estimator, and to investigate the asymptotic behavior of the Bayes optimal adaptive estimation scheme. First the optimal and the suboptimal approaches discussed in Chapter 2 were extended to the case of unknown transition matrix. Then conditions were investigated under which the constrained maximum likelihood estimate of the transition matrix converged in probability to the correct value of the transition matrix. The convergence of the a posteriori probabilities of the transition matrices given the past observations, and the convergence of the performance of the optimal adaptive scheme were also investigated. Finally a simulation example was presented to illustrate the convergence results.

In Chapter 4 we dealt with dynamical systems with interrupted observations. The interrupted observation mechanism was expressed in terms of a stationary two-state Markov chain whose transition probability matrix was assumed to be unknown and to belong to a finite set. The asymptotic behavior of the Bayes optimal adaptive estimation scheme was investigated analytically as in Chapter 3. Since extensive simulation results for this case are available in the literature, none were presented.

An alternate model for the unknown, time-varying parameters was considered in Chapters 5 and 6. Attention was confined to systems with unknown noise covariances. It was assumed that multiple bounds on the unknown covariances and their time derivatives were available. In Chapter 5 we dealt only with constant, unknown covariances; the primary purpose was to illustrate the concepts involved. A detection-estimation approach was proposed for state estimation and its asymptotic behavior was analyzed. The primary objective of the proposed approach was to reduce the pessimism of the standard minimax estimator for large observation records and large uncertainties, while retaining its desirable small-sample properties. A simulation example was presented which demonstrated some of the advantages of the proposed scheme over the standard minimax scheme and the Bayes optimal MMSE estimation scheme.

The approach of Chapter 5 was extended to the case of unknown time-varying noise covariances in Chapter 6. The extension was found to require excessive computational capabilities, therefore, an approximation to the proposed scheme was discussed. The feasibility of the approximation was demonstrated through a simulation example.

Finally, we indicate some research problems which should follow the work presented in this thesis. In Chapters 3 and 4 we assumed that the unknown transition probability matrix could only take one of a finite number of values. Extension of the convergence results of Chapters 3 and 4 to the case when the unknown transition matrix lies in a compact set should be investigated. Then one can use the standard methods of parameter optimization like gradient method, conjugate gradient method, etc., to find the maximum likelihood estimate, which obviates the need for computing the joint probability density of the observations for each and every candidate value of the transition matrix, resulting in a considerable saving in computations. A tacit assumption in the use of discrete-time system models in this thesis is that the jumps in the unknown parameters occur exactly at the sampling time instants. Use of continuous-time systems should therefore result in better models.

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APPENDIX A

PSEUDO-BAYES APPROXIMATION TO STATE ESTIMATION IN SWITCHING ENVIRONMENTS

In this appendix the pseudo-Bayes approximation to the optimal MMSE estimator of Section 2.2 is described. This approximation, due to Ackerson and Fu [1], is based on approximating Equations (2.4) and (2.5). It is assumed that $f(x_k|Z_k)$ is normally distributed with mean $\hat{x}(k|k)$ and covariance P(k), whereas, in truth, it is a sum of S^{k+1} separate Gaussian distributions. By making this assumption, (2.5) is expressed as

$$P(i_k=i|Z_k) = \frac{f(z_k|i_k=i,Z_{k-1})P(i_k=i|Z_{k-1})}{\sum_{\ell=1}^{S} f(z_k|i_k=\ell,Z_{k-1})P(i_k=\ell|Z_{k-1})} \tag{A1}$$

The value of $P(i_k=i|Z_{k-1})$ is available from $P(i_{k-1}=\ell|Z_{k-1})$, $\ell=1,2,..,S$ and transition matrix π. Since

$$f(x_k|Z_k) \sim N(\hat{x}(k|k),P(k)) \tag{A2}$$

we have

$$f(z_k|i_k=i,Z_{k-1}) \sim N\{C(A\hat{x}(k-1|k-1) + B\mu^{(i)}) + \upsilon^{(i)}, CM_i(k)C^T + R^{(i)}\} \tag{A3}$$

Furthermore (2.4) may be written as

$$\hat{x}(k|k) = \sum_{\ell=1}^{S} P(i_k=\ell|Z_k)\hat{x}_\ell(k|k) \tag{A4}$$

where $\hat{x}_\ell(k|k)$ is calculated from (2.9)-(2.12) by using $\hat{x}(k-1|k-1)$ as the previous estimate. The new covariance matrix P(k) is defined as

$$P(k) = \sum_{\ell=1}^{S} P(i_k=\ell|Z_k)P_\ell(k) \tag{A5}$$

where $P_{\ell}(k)$ is calculated as in (2.12). Notice that, unlike the optimal estimator case, now, given sequence $I_j(k)$, $\hat{x}_j(k|k)$ depends on π. The memory requirements are thus reduced to a constant independent of the number of steps the process runs.

APPENDIX B

RANDOM SAMPLING APPROACH TO STATE ESTIMATION IN SWITCHING ENVIRONMENTS

In this appendix the random sampling approximation to the optimal MMSE estimator of Section 2.2 is described. It is due to Akashi and Kumamoto [2]* and is based on a Monte Carlo method. The space Ω_k of the Markov chain state sequences $I_j(k)$, $j = 1,2,\ldots, S^{k+1}$, is regarded as a population and the state estimate is calculated with a relatively small number of sequences sampled at random from the population. Equation (2.4) may be rewritten as

$$x(k|k) = \hat{x}(k|k) = \Big(\sum_{I_j(k)} \hat{x}_j(k|k) f_k(I_j(k)) \Big) \Big/ \sum_{I_j(k)} f_k(I_j(k)) \qquad \text{(B1)}$$

where

$$f_k(I_j(k)) \triangleq \prod_{\ell=1}^{k} [f(z_\ell | I(\ell), Z_{\ell-1}) P(i_\ell | i_{\ell-1})] \qquad \text{(B2)}$$

Let the integer N be the number of sequences $I^{\nu}(k,k) = \{i_0^{\nu}, i_1^{\nu}, \ldots, i_k^{\nu}\}$, $\nu = 1,\ldots, N$, sampled by the procedure of Step 1) given below. For each integer ν, $1 \le \nu \le N$, let $\{\xi_0^{\nu}, \xi_1^{\nu}, \ldots, \xi_k^{\nu}, \ldots\}$ be a sequence of independent random numbers each of which is distributed with uniform distribution between 0 and 1. It is assumed that these random numbers are independent of $\{w_k\}$, $\{v_k\}$, x_o and $\{i_k\}$. The subsequence $\{\xi_\ell^{\nu}, 0 \le \ell \le k\}$ is used to sample the sequence $I^{\nu}(k,k)$ at the k-th stage. Now define sampling probabilities $\tilde{P}_k(i_\ell | I(\ell-1); Z_k)$, $\ell = 0,1,..,k$, satisfying the following conditions.

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*In [2] discussion is confined to switchings in measurement noise only. However, extension of their approach to a more general switching model is straightforward.

c1) For each ℓ, $\tilde{P}_k(i_\ell|I(\ell-1);Z_k)$ is a function of $I(\ell)$ and its functional form is determined either by Z_k or a subsequence of Z_k.

c2) With respect to the argument i_ℓ, $\tilde{P}_k(i_\ell|I(\ell-1);Z_k)$ is a probability on the set $\Omega \triangleq \{1,2,..,S\}$.

c3) Let Ω_k denote the Cartesian product of k sets Ω. The probability on Ω_k which is defined by

$$\tilde{P}_k(I(k);Z_k) = \prod_{\ell=0}^{k} \tilde{P}_k(i_\ell|I(\ell-1);Z_k) \tag{B3}$$

differs from zero whenever $f_k(I(k))$ differs from zero.

The minimum variance estimator $x(k|k)$, for a given $\pi = \pi_q$, is then calculated as follows:

Step 1: Suppose that the ℓ-long sequence $I^\nu(k,\ell-1) \triangleq \{i^\nu_{k,0},\ldots,i^\nu_{k,\ell-1}\}$ has been sampled. Then an $\alpha \in \{1,..,S\}$ is found to satisfy

$$\sum_{i_\ell=1}^{\alpha-1} \tilde{P}_k(i_\ell|I^\nu(k,\ell-1);Z_k) < \xi^\nu_\ell \leq \sum_{i_\ell=1}^{\alpha} \tilde{P}_k(i_\ell|I^\nu(k,\ell-1);Z_k) \tag{B4}$$

and the next state in the sequence is chosen such that

$$I^\nu(k,\ell) = \{I^\nu(k,\ell-1),i^\nu_{k,\ell}\} = \{I^\nu(k,\ell-1),\alpha\} \tag{B5}$$

If for each ν, $1 \leq \nu \leq N$, Step 1) is performed for $\ell = 0,1,..,k$, then the N sequences $I^\nu(k,k) = \{i^\nu_{k,0},\ldots,i^\nu_{k,k}\}$ can be sampled. In practical terms, (B4) and (B5) imply random sampling of the value of $i^\nu_{k,\ell}$ from the set Ω with probability $\tilde{P}_k(i_\ell|I^\nu(k,\ell-1);Z_k)$. Therefore, $I^\nu(k,k)$ is distributed with probability $\tilde{P}_k(I(k);Z_k)$ of (B3).

Step 2: The estimate $x(k|k)$ is approximated by $\tilde{x}(k|k)$:

$$\tilde{x}(k|k) = \frac{N^{-1}\sum_{\nu=1}^{N}[\hat{x}_\nu(k|k)f_k(I^\nu(k,k))/\tilde{P}_k(I^\nu(k,k);Z_k)]}{N^{-1}\sum_{\nu=1}^{N}[f_k(I^\nu(k,k))/\tilde{P}_k(I^\nu(k,k);Z_k)]} \tag{B6}$$

where $\hat{x}_\nu(k|k) = E\{x_k|Z_k,I^\nu(k,k)\}$. Akashi and Kumamoto [2] have shown that the following properties hold:

P1) For given Z_k, the numerator and the denominator of (B6) are the unbiased estimators of the numerator and the denominator of (B1), respectively. That is,

$$E\{N^{-1}\sum_{\nu=1}^{N}[\hat{x}_\nu(k|k)f_k(I^\nu(k,k))/\tilde{P}_k(I^\nu(k,k);Z_k)]|Z_k\} = \sum_{I_j(k)}\hat{x}_j(k|k)f_k(I_j(k)) \tag{B7}$$

and

$$E\{N^{-1}\sum_{\nu=1}^{N}[f_k(I^\nu(k,k))/\tilde{P}_k(I^\nu(k,k);Z_k)]|Z_k\} = \sum_{I_j(k)}f_k(I_j(k)) \tag{B8}$$

P2) Given Z_k, $I^\nu(k,k)$, $\nu=1,..,N$, are independent random variables. That is,

$$P(I^1(k,k),\ldots,I^N(k,k)|Z_k) = \prod_{\nu=1}^{N}\tilde{P}_k(I^\nu(k,k);Z_k) \tag{B9}$$

P3) Given Z_k, $\lim_{N\to\infty} N^{-1}\sum_{\nu=1}^{N}[f_k(I^\nu(k,k))/\tilde{P}_k(I^\nu(k,k);Z_k)] = f(Z_k)$ a.e. $\tilde{P}_k(I(k);Z_k)$. This follows from the law of large numbers and Equations (B2), (B8) and (B9).

P4) Let $J(Z_k,N) = E\{(x_k-\tilde{x}(k|k))^T Q(x_k-\tilde{x}(k|k))|Z_k\}$

and $J^o(Z_k) = E\{(x_k-x(k|k))^T Q(x_k-x(k|k))|Z_k\}$.

Then $J(Z_k,N) = J^o(Z_k) + O(N^{-1})$.

It remains to specify a procedure to determine $\tilde{P}_k(i_\ell|I(\ell-1);Z_k)$, $\ell=0,1,..,k$. In [2] the following method has been suggested:

$$\tilde{P}_k(i_\ell|I(\ell-1);Z_k) = P(i_\ell|I(\ell-1),Z_\ell) = \frac{f(z_\ell|I(\ell),Z_{\ell-1})P(i_\ell|i_{\ell-1})}{\sum_{i_\ell=1}^{S} [\text{numerator}]} \tag{B10}$$

It satisfies conditions c1), c2) and c3). An important consequence of such a definition is that, for each ℓ, we have

$$\tilde{P}_\ell(i_\ell|I(\ell-1);Z_\ell) = \tilde{P}_{\ell+1}(i_\ell|I(\ell-1);Z_{\ell+1}) = ... = \tilde{P}_k(i_\ell|I(\ell-1);Z_k) \tag{B11}$$

Furthermore

$$I^\nu(k,k-1) = I^\nu(k-1,k-1), \quad \nu = 1,2,..,N \tag{B12}$$

Therefore, the sequence $I^\nu(k-1,k-1)$ which is used to calculate $\tilde{x}(k-1|k-1)$ may again be used as the subsequence of $I^\nu(k,k)$, so that $I^\nu(k,k)$ at the k-th stage can be obtained by sampling only the k-th component $i^\nu_{k,k}$.

The storage and computation requirements are thus reduced to a constant (N filters) independent of the number of steps the process runs.

VITA

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